Stage 4 ★
Mixed Selection 1 – Solutions

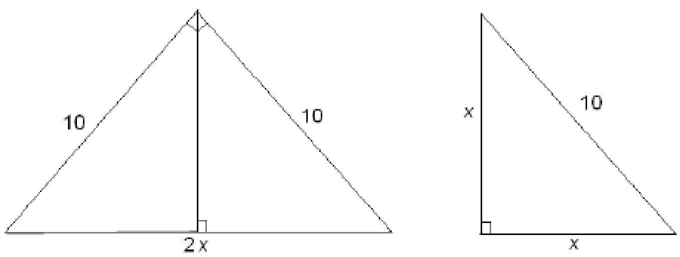
1. **Tetromino diagonal**
   By Pythagoras' theorem the length of the diagonal is $\sqrt{2^2+3^2}=\sqrt{13}$

2. **Out of the window**
   By Pythagoras Theorem, the diagonal of the window is 100 cm, which exceeds the length or breadth of all the sheets.

   So the first three pieces can go through the window either way and the 90 x 105 cm piece can also go through the window, provided the 90 cm edge goes first.

3. **Folding in half**
   By Pythagoras' Theorem, the hypotenuse of the original triangle is $\sqrt{200}=10\sqrt{2}$ cm, and hence the difference between the perimeters of the two triangles is $(10+10+10\sqrt{2})$-$(5\sqrt{2}+5\sqrt{2}+10)=10\text{cm}$

   Alternatively: let the length of the shorter sides of the new triangle be *x* cm, shown below. Then the perimeter of the original triangle is $(20+2x)$ cm and the perimeter of the new triangle is $(10+2x)\text{cm}$. Hence the difference between the perimeters of the two triangles is 10cm.



These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk).

4. Right angled possibilities

The hypotenuse is the longest side of the triangle, so the 5cm side cannot be the hypotenuse. This gives two possibilities:

a. The hypotenuse is 6cm, giving the other length as $\sqrt{6^2 - 5^2} = \sqrt{11}$

b. The 6cm side is the other short side, giving the hypotenuse as $\sqrt{6^2 + 5^2} = \sqrt{61}$

5. Rectangle Rearrangement

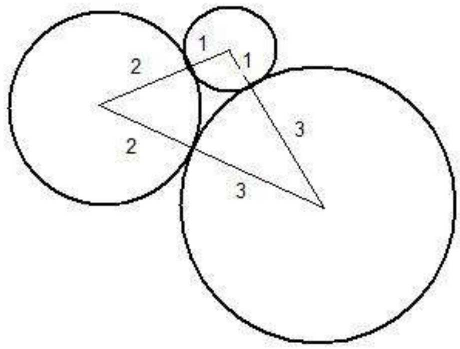
The two pieces will fit together to form a right-angled triangle which has a base 8 and height 6.

The length of the hypotenuse is $\sqrt{6^2 + 8^2} = 10$, so the perimeter of the triangle is 24.

Stage 4 ★★
Mixed Selection 1 – Solutions

1. Circle time

The triangle that joins up the centres of the circles has sides of length 3 cm, 4 cm, 5 cm. Since there is a right-angled triangle with sides 3, 4, 5, and since if two triangles share the lengths of their sides they are congruent, this angle must be right-angled. Therefore the length of the longer arc of the circle C_1 is $\frac{3}{4}\times 2\pi\times 1=\frac{3}{2}\pi$ cm.

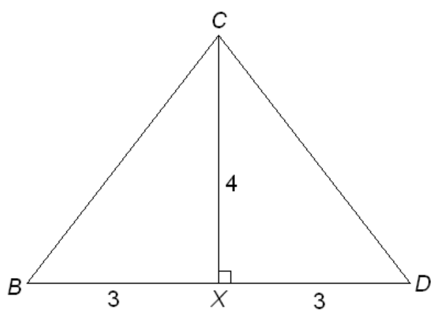


2. Unusual Polygon

The area of square *BDFG* is 6×6=36 square units.
So the total area of the three triangles *ABG*, *BCD* and *DEF* is also 36 square units. Since these triangles are congruent, each has an area of 12 square units.

The area of each triangle is $\frac{1}{2}\times \text{base} \times \text{height}$, and the base is 6 units and hence we have $\frac{1}{2}\times 6 \times \text{height} = 12$, so the height is 4 units.

Let *X* be the midpoint of *BD*. Then *CX* is perpendicular to the base *BD* (since *BCD* is an isosceles triangle).



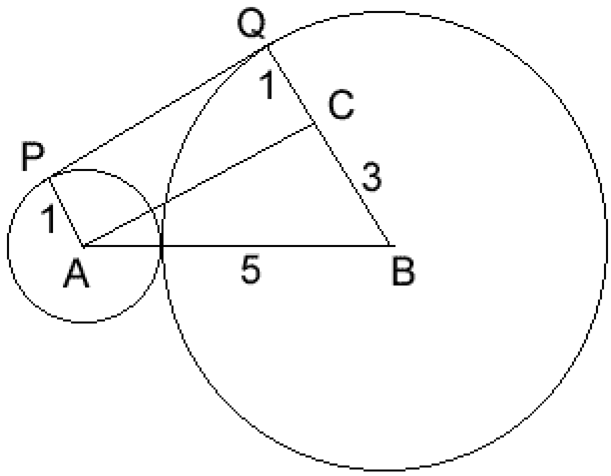
By Pythagoras' Theorem, $BC = \sqrt{3^2+4^2} = 5$

Therefore the perimeter of *ABCDEFG* is
$$6\times 5 + 6 = 36$$

These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk).

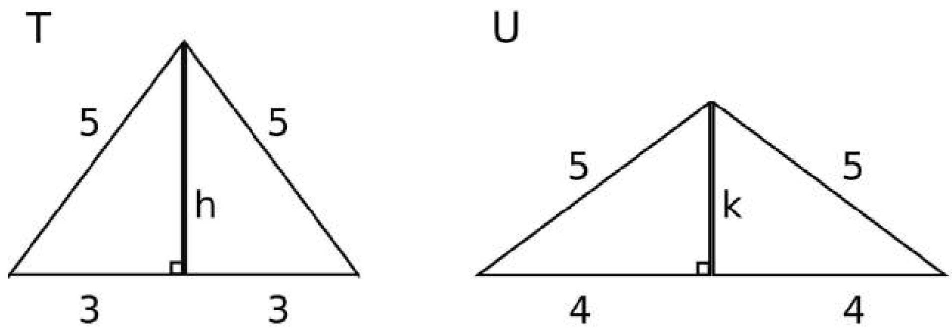
3. Common tangent

The diagram shows points A and B, the centres of the two circles, and C, on line BQ such that AC is parallel to PQ. Since ACQP is a rectangle, ACB is a right angled triangle. So By Pythagoras' Theorem, AC=4, which is also the length of PQ.



4. Triangular teaser

The diagram below shows isosceles triangles T and U. The perpendicular from the top vertex to the base divides an isosceles triangle into two congruent right-angled triangles as shown in both T and U. Evidently, by Pythagoras' Theorem, $h=4$ and $k=3$. So both triangles T and U consist of two 3, 4, 5 triangles and therefore have equal areas.

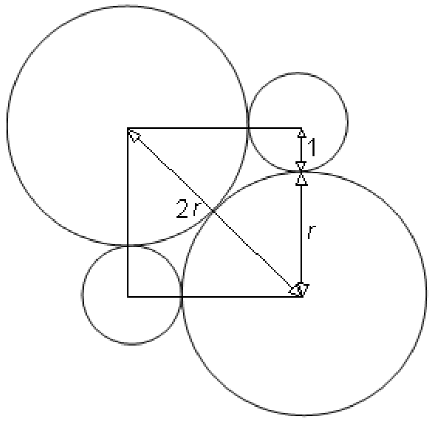


These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk).

Stage 4 ★★
Mixed Selection 2 – Solutions

1. Centre square

Let r be the radius of each of the larger circles.
The sides of the square are equal to $r + 1$, the sum of the two radii.
The diagonal of the square is $2r$.



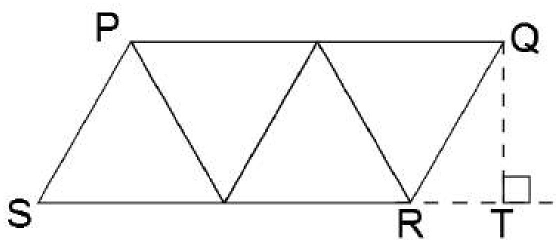
By Pythagoras, $(r+1)^2+(r+1)^2 = (2r)^2$
Simplifying gives: $(r+1)^2 = 2r^2$
so $r+1 = \sqrt{2}r$

Therefore $(\sqrt{2}-1)r = 1$
Hence $r = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$

2. Crane arm

In the diagram, T is the foot of the perpendicular from Q to the extension of SR. All of the angles in the equilateral triangles are 60°, so ∠QRT is also 60°. Then ΔQRT is a right-angled triangle, so using trigonometry we can show that the lengths of RT and QT are $\frac{1}{2}$ and $3\sqrt{2}$ respectively.
Applying Pythagoras' Theorem to ΔQST,

$$SQ^2 = ST^2 + QT^2 = 5^2 + \left(\sqrt{3}\right)^2 = 25 + 3 = 28$$



So the length of SQ is $\sqrt{28} = 2\sqrt{7}$ units.

These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk).

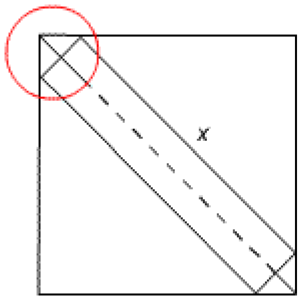
3. Walk the plank

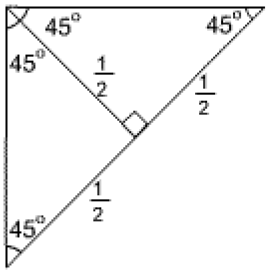
The figure shows the top left-hand corner of the complete diagram.
Note the symmetry which leads to the three measurements of $\frac{1}{2}$.
Thus the diagonal of the square can be divided into three portions of lengths:

$\frac{1}{2}, x$ and $\frac{1}{2}$ respectively.

The length of the diagonal is $\sqrt{10^2 + 10^2} = \sqrt{200} = 10\sqrt{2}$

So $x = 10\sqrt{2} - 1$.



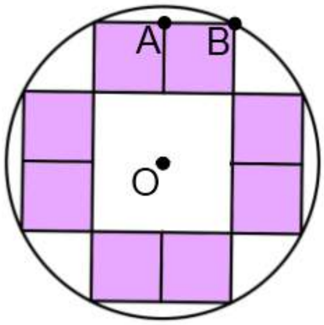


4. Indigo interior

Let the centre of the circle be *O* and let *A* and *B* be corners of one of the shaded squares, as shown.

The circle has area π square units, so *OB* is 1 unit long.

Let the length of the side of each of the shaded squares be *x* units.
By Pythagoras, $OB^2 = OA^2 + AB^2$; that is $1^2 = (2x)^2 + x^2$.
So $5x^2 = 1$.
Now the total shaded area is $8x^2 = \frac{8}{5}$ square units.

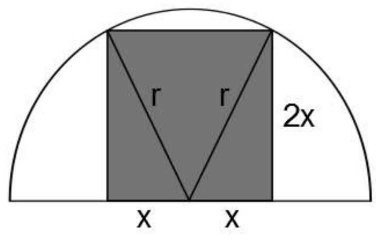


These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk).

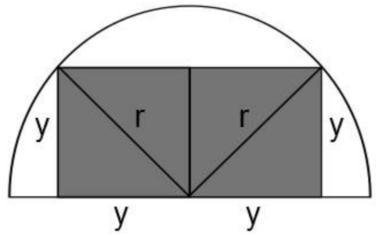
Stage 4 ★★★
Mixed Selection 1 – Solutions

1. One or two

Let the radius of each semicircle be r.
In the top diagram, let the side-length of the square be $2x$.
By Pythagoras' Theorem, $x^2 + (2x)^2 = r^2$ and so $5x^2 = r^2$.
So this shaded area is $(2x)^2 = 4x^2 = \frac{4r^2}{5}$.



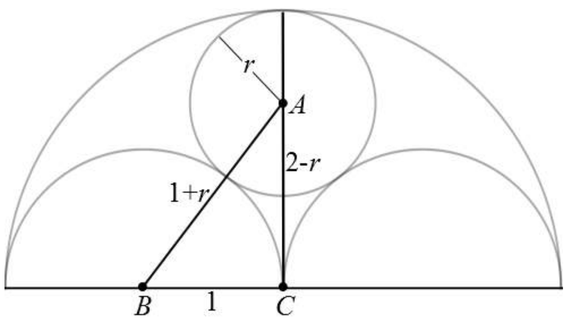
In the bottom diagram, let the side-length of each square be y. Then by Pythagoras' Theorem, $y^2 + y^2 = r^2$ and so $2y^2 = r^2$.
So this shaded area is r^2.



Therefore the ratio of the two shaded areas is 4:5.

2. Salt's Mill

Let *A* be the centre of the circle, *B* be the centre of the left-hand semicircle and *C* be the centre of the large semicircle. Then we can mark their lengths below:



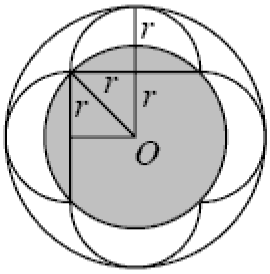
By Pythagoras' Theorem:
$(1+r)^2=1^2+(2-r)^2$
$1+2r+r^2=1+4-4r+r^2$
$6r=4$

So the radius is 2/3 of a metre.

These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk).

3. Oh so circular

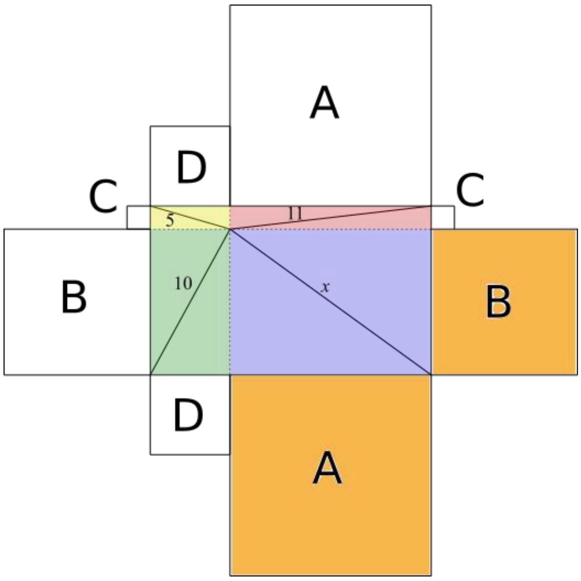
From the symmetry of the figure, the two circles must be concentric. Let their centre be O. Let the radius of the semicircles be r. Then the radius of the outer circle is $2r$ and, by Pythagoras' Theorem, the radius of the inner shaded circle is $\sqrt{r^2+r^2}$, that is $\sqrt{2}r$.



So the radii of the two circles are in the ratio $\sqrt{2}:2$, and hence the ratio of their areas is $2:4=1:2$.

4. Distance to the corner

In the diagram below, the courtyard has been split into 4 rectangles whose corners are at the well. The squares drawn on are for the diagrammatic representation of Pythagoras' theorem, so that the area of shaded square A added to the area of shaded square B is equal to x^2, because x is the hypotenuse of the triangle in the blue rectangle.



The square of the hypotenuse in the red rectangle is A+C, and the square of the green rectangle is B+D. Adding these together gives A+B+C+D, which is too much by C+D. But the square of the hypotenuse in the yellow rectangle is C+D.

So A+B $=11^2+10^2-5^2=196$.

Then $x^2 = 196$, so $x = 14$.

A fuller solution is available at: https://nrich.maths.org/12899/solution

These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk).

Stage 4 ★★★
Mixed Selection 1 - Solutions

1. Symmetric angles

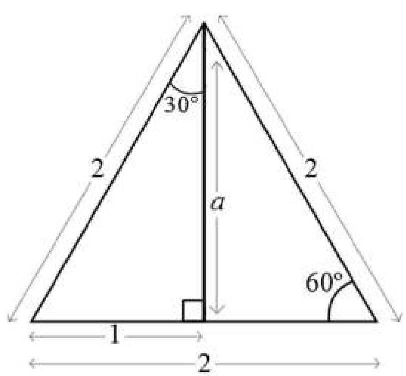
As the figure has rotational symmetry of order 4, $ABEF$ is a square.

Area $ABEF = 4 \times$ area of $\triangle BDA = 4 \times \frac{1}{2} BD \times D \quad = 2(BD)^2 = 24\text{cm}^2$, so $BD = \sqrt{12}cm = 2\sqrt{3}cm$.

As $ABEF$ is a square, $\angle ABD = 45°$ so $\angle CBD = 45° - 15° = 30°$.

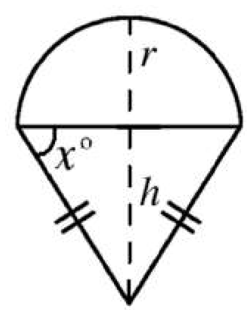
Since $\tan 30° = \frac{CD}{BD} = \frac{CD}{2\sqrt{3}}$, we have $CD = 2\sqrt{3}\tan 30\,\text{cm}$.

Now consider the following equilateral triangle with side lengths 2:



The vertical line is perpendicular to the base-line and so bisects both the angle at the top vertex and the base-line. Consider the left right-angled triangle. Pythagoras' theorem gives $a = \sqrt{3}$ and then we have $\tan 30 = \frac{1}{a} = \frac{1}{\sqrt{3}}$.
Therefore $CD = 2\sqrt{3}\tan 30 = 2\text{cm}$.

2. Ice cream tangent



Let the radius of the circle be r and let the perpendicular height of the triangle be h. Then

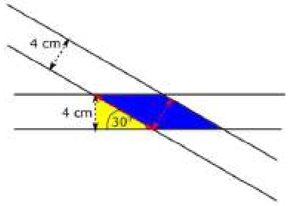
$$\tan x = \frac{h}{r}$$

Now, the area of the semicircle is $\frac{1}{2}\pi r^2$ and the area of the triangle is $\frac{1}{2} \times 2r \times h$. This gives $rh = \frac{1}{2}\pi r^2$, so $\frac{h}{r} = \frac{\pi}{2}$.

These problems are adapted from UKMT (ukmt.org.uk) and SEAMC (seamc.asia) problems.

3. Overlapping ribbons

In this diagram, the length and perpendicular 'height' of the parallelogram formed by the overlap are marked in red, and a helpful triangle is coloured yellow.



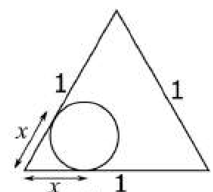
The 'height' of the parallelogram is 4 cm and the length of the parallelogram is equal to the hypotenuse of the yellow triangle. This hypotenuse can be found using trigonometry.

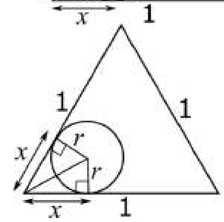
$$\sin(30) = 4\text{hyp} \Longrightarrow \sin(30) \times \text{hyp} = 4 \Longrightarrow \text{hyp} = 4\sin(30) = 8$$

So the length of the parallelogram is 8 cm and its 'height' is 4 cm, so its area is $8 \times 4 = 32\,cm^2$.

A fuller solution is available at: https://nrich.maths.org/12761/solution

4. The roller and the triangle





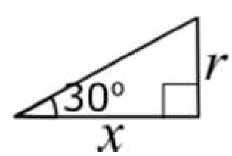
The circle won't fit into the corners of the triangle, so it won't touch some of the parts of the perimeter of the triangle. The parts that it won't touch are the parts that Clare can't paint with the roller, and they are labelled x on the diagram.

Using the radius of the circle, two right-angled triangles can be made which contain x, as shown below (they are right-angled because the angle between the tangent and the radius is 90^o).

The radius of the circle can be found using the formula circumference $= \pi \times 2r$, so $1 = 2\pi r$, so $r = \frac{1}{2\pi}$.

We can use trigonometry to find out the unknown length x. Since the two right-angled triangles drawn share all three sides, they are congruent, so the 60^o angle of the triangle must be bisected to give two 30^o angles.

This gives the triangle shown. $\tan(30) = \frac{r}{x}$, and since $r = \frac{1}{2\pi}$ and $\tan(30) = \frac{1}{\sqrt{3}}$, we get that:



$$\frac{1}{\sqrt{3}} = \frac{\frac{1}{2\pi}}{x} \Rightarrow \frac{x}{\sqrt{3}} = \frac{1}{2\pi} \Rightarrow x = \frac{\sqrt{3}}{2\pi}$$

So the circle will touch all of the perimeter of the triangle, which is 3, except for x twice at each corner. So the circle will touch $3 - 6x = 3 - 6\left(\frac{\sqrt{3}}{2\pi}\right) = 3 - \frac{3\sqrt{3}}{\pi}$.

A fuller solution is available at: https://nrich.maths.org/12788/solution

These problems are adapted from UKMT (ukmt.org.uk) and SEAMC (seamc.asia) problems.