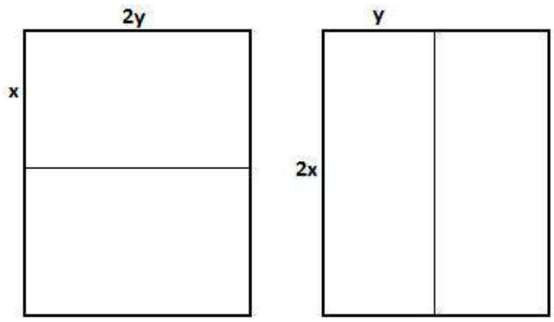
Stage 3 ★
Mixed Selection 1 – Solutions

1. **3x3 areas**
All but D have an area of 3 square units. D has an an area half a unit less than the other shapes.

2. **Mid-point area**
M is the midpoint of the side of the rectangle, so RM has length 2 units.

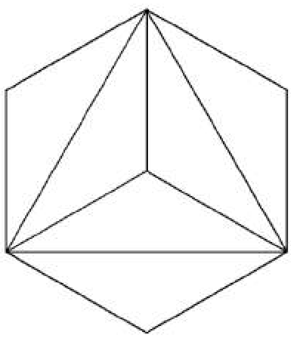
$$\begin{aligned}\text{Area of triangle PRM} &= \frac{1}{2} \times \text{base} \times \text{height}\\ &= \frac{1}{2} \times 2 \text{ units} \times 3\text{units}\\ &= 3 \text{ square units}\end{aligned}$$

3. **Rectangle cutting**
Tom and Jerry must each cut their piece of paper in half. Suppose the sides of the original piece of paper have length $2x$ and $2y$, with $x \geq y$.



Then $2x + 4y = 40$ and $4x + 2y = 50$ which implies that $6x + 6y = 90$, therefore the perimeter of the original piece, $4x + 4y = 60$.

4. **Hexa-split**
All the areas of the three figures are equal (X = Y = Z).



The shaded area in Y is clearly half the area of the hexagon and the diagram below demonstrates why the shaded areas in X and Z are also both equal to half the area of the hexagon.

Stage 3 ⋆
Mixed Selection 2 – Solutions

1. **Shaded End**
There are four faces that are the same size as the unshaded ones, and two that are the same as the shaded face.
Since each unshaded face is four times the size of the shaded one, the surface area of the cuboid is $4 \times 4 + 2 \times 1 = 18$ times the area of the shaded face.

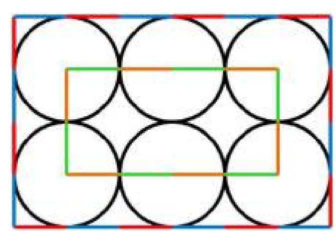
Since this total area is $72cm^2$, the shaded face has area $72 \div 18 = 4cm^2$
Each unshaded face therefore has area $4 \times 4 = 16cm^2$

2. **Square ratio**
Let the length of a short side of a rectangle be x and the length of a long side be y. Then the whole square has side of length $(y + x)$, whilst the small square has side of length $y - x$.

As the area of the whole square is four times the area of the small square, the length of the side of the whole square is twice the length of the side of the small square.

Therefore $y + x = 2(y - x)$, i.e., $y = 3x$ so $x: y = 1: 3$.

3. **Six circles**
The small rectangle consists of 12 of the radii of the circles, each connecting a point of contact to the centre of the relevant circle. These are shown in green and orange in the diagram on the right.



Since this has a total length of 60cm, each radius is of length $60\text{cm} \div 12 = 5\text{cm}$.

The large rectangle can also be broken down into segments of this length. These are shown in blue and red on the diagram. There are 20 of these, so the perimeter of the large rectangle is $5\text{cm} \times 20 = 100\text{cm} = 1\text{m}$.

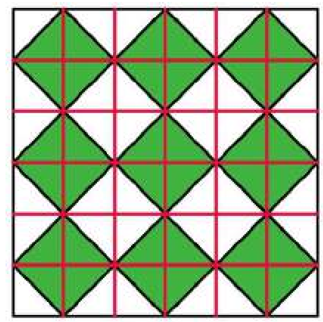
These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk)

4. Squares in a square

There are a number of ways of solving this problem. One method is to count the unshaded squares in the diagram. There are 4 complete squares, 8 half squares and 4 quarter squares, which is a total of $4 \times 1 + 8 \times \frac{1}{2} + 4 \times \frac{1}{4} = 4 + 4 + 1 = 9$.

Therefore there are nine shaded squares and nine unshaded squares, so half the area is shaded.

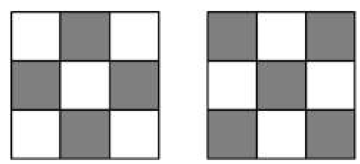
Alternatively, consider dividing the square up into the smaller red squares, shown in the diagram to the right. Each of the red squares is divided into two halves, one of which is shaded.



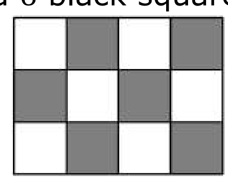
This means that half of the complete shape is shaded.

5. Chequered cuboid

Consider the different faces of this cuboid. The faces on the ends of the cuboid will look like these diagrams on the right. Between them they have 9 black squares and 9 white squares.



The other four faces all look like the diagram below. This contains 6 white squares and 6 black squares.



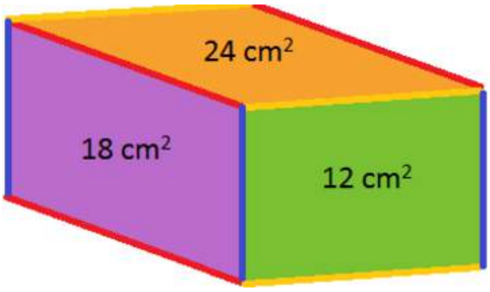
Therefore, between the four faces there are $6 \times 4 = 24$ white squares and $6 \times 4 = 24$ black squares.

Therefore, overall there are $9 + 24 = 33$ white squares and $9 + 24 = 33$ black squares, so half of the squares are coloured black.

Stage 3 ★
Mixed Selection 3 - Solutions

1. **Cuboid faces**

From the above diagram we can see that:



- the length of the yellow edge multiplied by the length of the blue edge must be 12.

-the length of the red edge multiplied by the length of the blue edge must be 18.

-the length of the yellow edge multiplied by the length of the red edge must be 24.

We can start by thinking of two numbers that multiply to get 12, and then choose the right number to make the purple face have an area of 18 square centimetres. Then we can check whether those numbers give us the correct area for the orange face.

If the yellow edge is 6 cm long and the blue edge is 2 cm long, then the red edge must be 9 cm long, because $2\times9 = 18$. However, then the orange face would have area $6\times9 = 54$ square centimetres, which is not right.

If the yellow edge is 2 cm long and the blue edge is 6 cm long, then the red edge must be 3 cm long, because $6\times3 = 18$. However, then the orange face would have area $2\times3 = 6$ square centimetres, which is not right.

If the yellow edge is 4 cm long and the blue edge is 3 cm long, then the red edge must be 6 cm long, because $3\times6 = 18$. Then the orange face would have area $4\times6 = 24$ square centimetres, which is right.

So the edges of the cuboid are 3 cm, 4 cm and 6 cm long, which means the volume of the cuboid is $3\times4\times6 = 72\ cm^3$.

A fuller solution is available at: http://nrich.maths.org/12466/solution

2. Strawberries and peas

The length of the pea bed was increased by $3\mathrm{m}$,so the length of the strawberry patch was reduced by $3\mathrm{m}$. This reduced the area by $15\mathrm{m}^2$. Therefore, the rectangle that was transferred had length 3m and area $15\mathrm{m}^2$. The width of the rectangle is therefore $15 \div 3 = 5$, so the garden is $5\mathrm{m}$ wide.

The new pea bed is a square. As it is $5\mathrm{m}$ wide, it is also $5\mathrm{m}$ long. This is after it had a side increased by $3\mathrm{m}$, so originally it was $5\mathrm{m} \times 2\mathrm{m}$. The area of this is $5\times 2 = 10\mathrm{m}^2$.

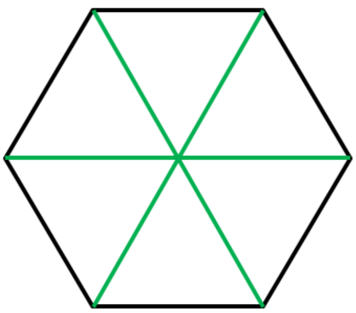
3. Giant Rubik's cube

The outside cubes enclose a 10 x 10 x 10 cube, so the number of visible cubes is the total cubes minus these. This is $12^2 - 10^3 = 728$.

A fuller solution is available at: https://nrich.maths.org/12840/solution

4. Star in a hexagon

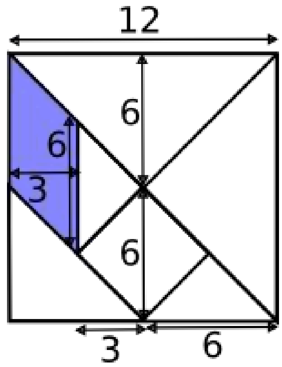
The diagram on the right demonstrates how each of the small hexagons can be split into six of the small triangles. The shaded area is therefore the equivalent of 12 of the small triangles. The whole large hexagon is the equivalent of
$7\times 6 + 12 = 54$ small triangles.



Therefore the shaded area is $\frac{12}{54} = \frac{2}{9}$ of the whole hexagon.

These problems are adapted from UKMT (ukmt.org.uk) and SEAMC (seamc.asia) problems.

Stage 3 ★★
Mixed Selection 1 – Solution



1. Tangram Area

The parallelogram has base 6cm and height 3cm, so the area is

$$6\text{cm} \times 3\text{cm} = 18\text{cm}$$

2. Triangles' triangle

The perimeter of the large triangle is 24cm, so it has side of length 8cm, so each small triangle has side of length 1cm. So each small triangle has perimeter 3cm.

$$\begin{aligned}\text{Total length of black lines} &= \text{Total perimeter of shaded triangles}\\ &= \text{Nubmer of shaded triangles} \times 3\text{ cm}\\ &= 27 \times 3\text{cm}\\ &= 81\end{aligned}$$

3. L-emental

The perimeter of the shape is $4\ \times \text{length}$, so the original rectangles are $10\ cm$ long.

There is insufficient information to be able to say anything about the width - except that it is less than $10\ cm$.

4. Guillotine

Let AB and AD be of length b and h respectively. Then the area of $ABCD = bh$ and the area of $ABQP = \frac{1}{2}b\left(\frac{h}{2} + \frac{h}{3}\right) = \frac{5bh}{12}$.

Thus the required ratio is $\frac{5}{12}$.

5. Margins

The area of the page inside the margins is $26cm\ \times 36cm\ =\ 936cm^2$.

So the percentage of the page occupied by the margins is

$$\frac{264}{1200} \times 100\%\ =\ 22\%.$$

Stage 3 ★★
Mixed Selection 2 – Solutions

1. Intersecting squares

Each of the overlapping areas contributes to the area of exactly two squares. So the total area of the three squares is equal to the area of the non-overlapping parts of the squares plus twice the total of the three overlapping areas,
i.e. $(117 + 2(2 + 5 + 8))\text{cm}^2 = (117 + 30)\text{cm}^2 = 147\text{cm}^2$.

So the area of each square is $(147 \div 3)\text{cm}^2 = 49\text{cm}^2$. Therefore the length of the side of each square is 7cm.

2. Cubic masterpiece

Let the smaller cubes have side length 1unit. So the original cube had side of length 3unit and, as a cube has six faces, it had a surface area of $6 \times (3\text{units} \times 3\text{units}) = 54\text{units}^2$, all of which was painted blue.

The total surface are of the 27 small cubes is
$27 \times 6\text{units}^2 = 162\text{units}^2$.

So the required fraction is $\frac{54\text{units}^2}{162\text{units}^2} = \frac{1}{3}$.

3. Cubes on a cube

The shape consists of 7 cubes, and has a total volume of 875cm^3. Each cube therefore has volume $875\text{cm}^3 \div 7 = 125\text{cm}^3$.

Therefore, the side length of the cube is $\sqrt[3]{125\text{cm}^3} = 5\text{cm}$, so each face has area $(5\text{cm})^2 = 25\text{cm}^2$.

Each of the outer cubes has five of its faces showing, so there are $6 \times 5 = 30$ faces showing altogether. These have a total area of $30 \times 25\text{cm}^2 = 750\text{cm}^2$.

These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk)

4. Sideways ratio

Let the sides of the rectangle, in cm, be $4x$ and $5x$.

Then the area of the square is $4x \times 5\text{cm}^2 = 20x^2\text{cm}^2$. So $20x^2 = 125$, that is $x^2 = 254$. Therefore $x = \pm\frac{5}{2}$, but x cannot be negative so $x = 52$ and so the sides of the rectangle are 10cm and 12.5cm.

Hence the rectangle has perimeter 45cm.

5. Exactly three-quarters

The area of the rectangle is 48cm^2, so the unshaded area is 12cm^2.

Therefore taking the two unshaded triangles together:

$$\left(\frac{1}{2} \times x \times 2\right) + \left(\frac{1}{2} \times (6 - x) \times 8\right) = 12$$

which means that $x + 24 - 4x = 12$, so $x = 4$.

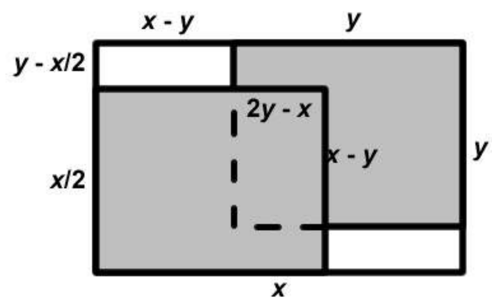
Stage 3 ★★
Mixed Selection 3 – Solutions

1. Double cover

We start by placing the two halves on the table so that the whole table is covered. Then we move them to overlap as in the question.

The area of the table covered by the paper has decreased by the two uncovered areas. The area covered by the paper has decreased by the amount covered twice. These two areas must both be the same, so the ratio must be $1:1$.

Alternatively, let the sheet of paper have length x and width y. Then the uncovered area consists of two congruent rectangles of length $x - y$ and width $y - 12x$. So the uncovered area is $2(x - y)(y - 12x)$, that is, $(x - y)(2y - x)$.



The area covered twice is a rectangle of length $y - (x - y)$, that is, 2y−x, and width $12x - (y - 12x)$, that is, $(x - y)$. So the area covered twice is also $(x - y)(2y - x)$.

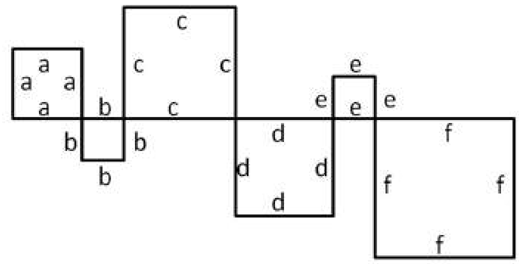
2. Open the box

The original square has area $180 + 4 \times 4 = 196$, so it has side length 14cm. Therefore, the dimensions of the box are $10 \times 10 \times 2$. Hence, the volume is 200cm^3.

Alternatively, the base of the open box is a square. Let its side be of length x cm. Then the total surface area of the box in cm^2 is $x^2 + 4 \times 2x = x^2 + 8x$. Hence, $x^2 + 8x = 180$, that is $x^2 + 8x - 180 = 0$. Therefore, $(x + 18)(x - 10) = 0$ which gives $x = -18$ or $x = 10$.

As x must be positive, it must be the case that $x = 10$. Now the open box has dimensions $10\text{ cm} \times 10\text{ cm} \times 2\text{ cm}$. So its volume is 200cm^3.

3. Line of squares



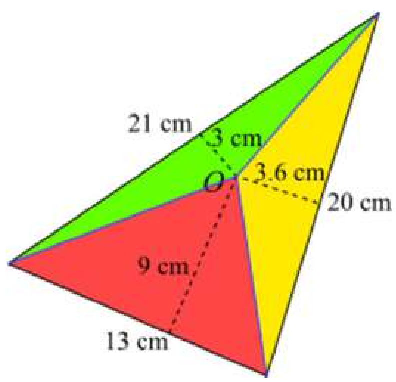
Each square has exactly $\frac{1}{4}$ of its perimeter as part of the line, and together these form the whole line. This means the total perimeter is $4 \times 24\text{cm} = 96\text{cm}$.

Algebraically, the diagram on the left shows that the line has length $a + b + c + d + e + f = 24\text{cm}$. The total of the perimeters is then:

$$4a + 4b + 4c + 4d + 4e + 4f = 4(a + b + c + d + e + f)$$
$$= 4 \times 24\text{cm}$$
$$= 96\text{cm}$$

4. Scalene area

Drawing lines from O to each vertex splits the triangle into three smaller triangles, as shown below.



For each smaller triangle, the 'base' and 'height' are shown (the 'bases' are the sides of the original triangle, which are not horizontal). So, using $Area = \frac{1}{2} base \times height$, the areas of the smaller triangles can be found:

Red triangle area $\frac{1}{2} \times 13 \times 9 = 58.5\ cm^2$

Yellow triangle area $\frac{1}{2} \times 20 \times 3.6 = 36\ cm^2$

Green triangle area $\frac{1}{2} \times 21 \times 3 = 31.5\ cm^2$

So the total area of the triangle is

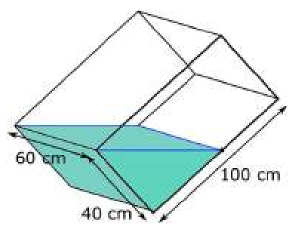
$58.5cm^2 + 36cm^2 + 31.5cm^2 = 126cm^2$.

A fuller solution is available at: https://nrich.maths.org/12760/solution

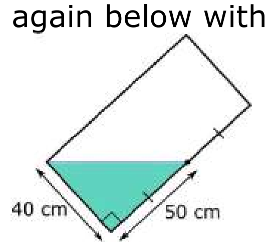
These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk)

Stage 3 ★★
Mixed Selection 4 - Solutions

1. **Tilted tank**
The water in the tank is in the shape of a triangular prism, as shown, so its volume can be found using length × area of cross-section.

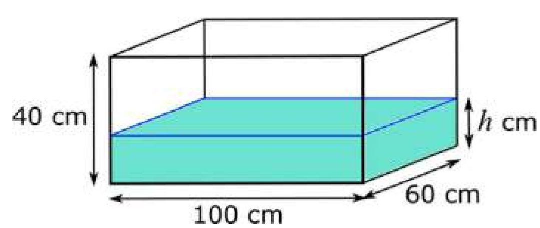


The cross-section is the triangle on the original diagram, shown again below with the sides labelled.



Because the two sides which are labelled are perpendicular, they can be considered to be the base and height of the triangle, so its area is $\frac{1}{2} \times 40 \times 50 = 1000\ cm^2$. So the volume of water in the tank is $60 \times 1000 = 60000\ cm^3$.

When the tank is resting on its base, as shown below, the volume of water in the tank is given by $100 \times 60h = 6000h\ cm^3$.

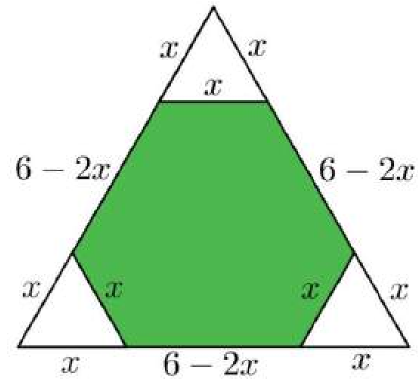


But the volume of water in the tank must also still be 60000 cm3, so $6000h = 60000$. So h must be 10.
So the height of the water is 10 cm.
A fuller solution is available at: http://nrich.maths.org/12782/solution

These problems are adapted from UKMT (ukmt.org.uk) and SEAMC (seamc.asia) problems.

2. Corner cut

Suppose the small equilateral triangles have sides of length x cm.
Then, each of these has perimeter $3x$, and there are three of these, so the total perimeter is $3 \times 3x = 9x$.



The hexagon has three sides where the triangles have been cut from, each of length x. The other sides are the sides of the larger triangle, less two sides of the smaller triangles. They, therefore, have length $6 - 2x$. There are three of each of these, so the perimeter of the hexagon is $3x + 3(6 - 2x)$.

Since the hexagon has a perimeter equal to the sum of the three small triangles, we get: $9x = 3x + 3(6 - 2x)$
Then, expanding the bracket gives: $9x = 3x + 18 - 6x$.
Collecting like terms: $9x = 18 - 3x$.
Adding $3x$ to each side: $12x = 18$.
Dividing by 6: $x = 1.5$.

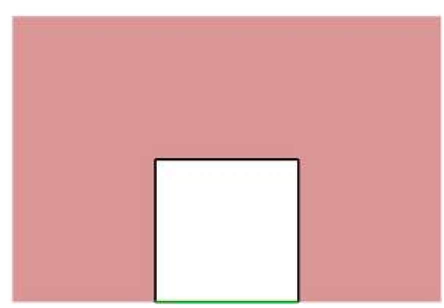
Therefore, the small equilateral triangles have sides of length 1.5cm.

3. Leaning over

The triangle HIJ has a base (HI) of 4cm, the same as the side of the square.
The area of the square is $4 \times 4 = 16\text{cm}^2$, so this is also the area of the triangle.
If the perpendicular height of HIJ is h, then $12 \times 4 \times h = 16$. This means $2h = 16$, so $h = 8\text{cm}$.
J is therefore 8cm above the top of the square, which is 4cm above the base, making a total of 12cm.

4. Christmas cut out

The original rectangle had a perimeter of $2 \times 30 + 2 \times 40 = 140cm$.

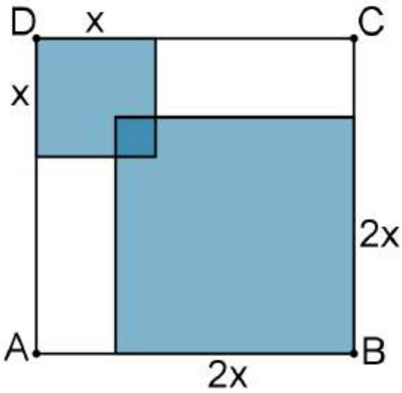


Each of the new squares that is cut out adds three sides of 5cm each (shown in black on the diagram), and removes one portion of length 5cm (shown in green).
This means the overall effect of cutting out each square is to increase the perimeter by 10cm.
Since are ten squares, the overall increase is $10 \times 10cm = 100cm$.
Therefore the final perimeter is $140cm + 100cm = 240cm$.

These problems are adapted from UKMT (ukmt.org.uk) and SEAMC (seamc.asia) problems.

Stage 3 ★★★
Mixed Selection 1 – Solutions

1. Four square



The large square has area $196 = 14^2$, so it has side-length 14. The ratio of the areas of the inner squares is $4:1$, so the ratio of their side-lengths is $2:1$.

Let the side-length of the larger inner square be $2x$, so that of the smaller is x. The figure is symmetric about the diagonal AC and so the overlap of the two inner squares is also a square which therefore has side-length 1.

Thus the vertical height can be written as $x + 2x - 1$.
Hence, $3x - 1 = 14$ and so $x = 5$

Therefore the small shaded square has an area of $5 \times 5 = 25$ and the larger shaded square has an area of $10 \times 10 = 100$

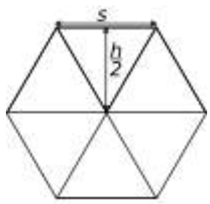
Therefore the total shaded area is $25 + 100 - 1 = 124$

Note also that the two unshaded rectangles have side-lengths $14 - 2x = 4$ and $14 - x = 9$

So the total unshaded area is $36 \times 2 = 72$ and therefore the total shaded area is $196 - 72 = 124$

2. Triangle in a hexagon

Call the length of one side of the hexagon s and the height of the hexagon h. So the area of the shaded triangle is $\frac{1}{2} \times s \times h$.

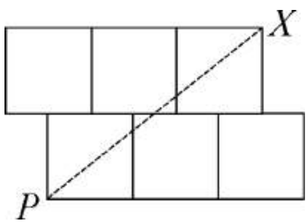


Now divide the hexagon into 6 equilateral triangles:

so the area of the hexagon is $6 \times 12 \times s \times \frac{h}{2} = 3 \times \frac{1}{2} \times s \times h$, or $3 \times \text{Shaded area}$. So the shaded area is 13 of the area of the whole hexagon.

3. Pile driver

The small square on top will be in the upper half of the divided figure. Now consider the figure formed by moving this square to become an extra square on the left of the second row, as shown.

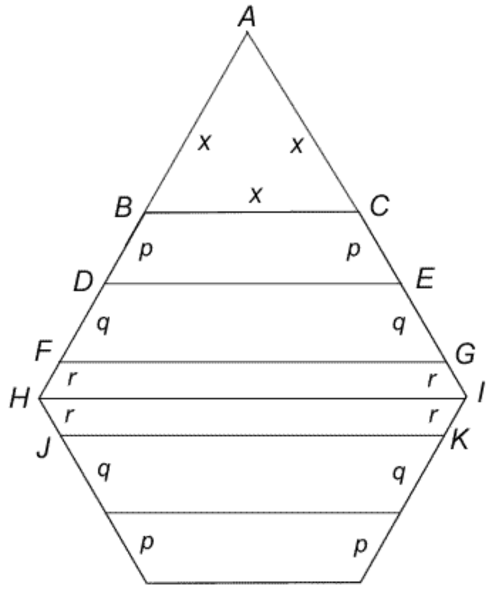


It may now be seen from the rotational symmetry of the figure that the line PX splits the new figure in half- with that small square in the upper half.

Therefore the line which halves the area of the original figure cuts the edge XY at X.

4. Hexagon slices

Each exterior angle of a regular hexagon is $\frac{360°}{6} = 60°$, so when sides HB and IC are extended to meet at A , an equilateral



triangle, ABC is created.

Let the sides of this triangle be of length x. As BC, DE and FG are all parallel, triangles ABC, ADE and AFG are all equilateral.
So, $DE = DA = p + x$; and $FG = FA = q + p + x$.

The perimeter of trapezium
$BCED = x + p + x + 2p = 2x + 3p$;

The perimeter of trapezium

$$\begin{aligned} DEGF &= (p + x) + (q + p + x) + 2q \\ &= 2x + 2p + 3q \end{aligned}$$

The perimeter of hexagon
$FGIKJH = 2((q + p + x) + 2r) = 2x + 2p + 2q + 4r$.

So, $2x + 3p = 2x + 2p + 3q$; hence $p = 3q$.
Also $2x + 2p + 3q = 2x + 2p + 2q + 4r$; hence $q = 4r$.
So, $p : q : r = 12r : 4r : r = 12 : 4 : 1$.

Stage 4 ★
Mixed Selection 1 - Solutions

1. Circle in a semicircle

Let the radius of the circle be r. This implies that the radius of the semicircle is $2r$. The area of the semicircle is $\frac{1}{2} \times \pi \times (2r)^2$, which is twice the area of the small circle.

2. Four parts

The circumference of a circle with radius r is $2\pi r$.

Each piece has a quarter of the circle with radius 4 together with two semicircular arcs of radius 2.

The arc of the large circle has length $\frac{1}{4} \times 2 \times \pi \times 4 = 2\pi$.
The semicircular arcs have length $\frac{1}{2} \times 2 \times \pi \times 2 = 2\pi$.
Therefore the overall perimeter is 6π.

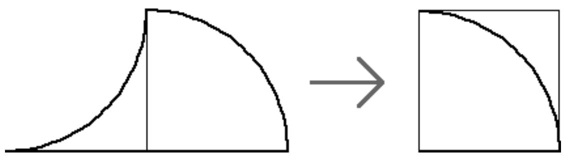
3. Smile

If the semicircle with diameter PQ is rotated through $180°$ about Q, the new shape formed has the same area as the original shape. It consists of a semicircle of diameter $6cm$ and a semicircle of diameter $2cm$.

So the area of the original shape is $\frac{1}{2} \times \pi \times 3^2 + \frac{1}{2} \times \pi \times 1^2 = 5\pi\ cm^2$.

4. Arc area

As the diagram shows, the figure may be cut into two parts that fit together to form a square measuring 8cm x 8cm.

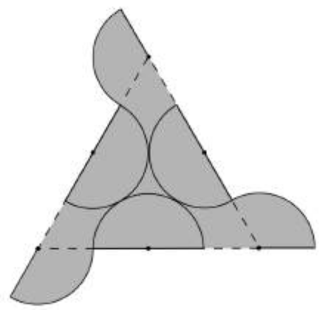


These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk)

Stage 4 ★
Mixed Selection 2 - Solutions

1. **Penny farthing**
As the ratio of the radii is $3:4$ the number of revolutions made by the larger wheel is $120000 \times \frac{3}{4} = 90000$

2. **Tadpoles**
The length of the side of the triangle is equal to four times the radius of the arcs. So the arcs have radius $2 \div 4 = 1/2$. In the diagram above, three semicircles have been shaded dark grey. The second diagram shows how these semicircles may be placed inside the triangle so that the whole triangle is shaded.



Therefore, the difference between the area of the shaded shape and the area of the triangle is the sum of the areas of three sectors of a circle. The interior angle of an equilateral triangle is $60°$, so the angle at the centre of each sector is $180° - 60° = 120°$.

Therefore, each sector is equal in area to one-third of the area of a circle. Their combined area is equal to the area of a circle of radius $1/2$. So the required area is $\pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$

3. Square Flower

The square has side length 2, so each of the circles has radius $\frac{1}{2}$.

The circumference of a single circle, $2\pi r$, is therefore π.

The shaded region is bounded by 4 halves of a circle and 4 quarters of a circle.

Therefore the length of the perimeter of the shaded region is

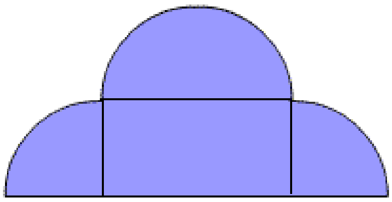
$$\left(4\ \times \frac{1}{2} + 4 \times \frac{1}{4}\right)\pi = 3\pi$$

4. Crazy shading

The middle sized square passes through the centres of the four circles. Each side of the middle sized square together with the edges of the outer square creates a right angled isosceles triangle with angles of 45∘. Thus the angles these sides make with the inner square are also 45∘. Each side of the middle sized square bisects the area of the circle. The inner half of that circle is made up of two shaded segments with angles of 45∘ which together are equal in area to the unshaded right angled segment. Thus the total shaded area is exactly equal to the area of the inner square and hence equal to one quarter of the area of the outer square.

Stage 4 ★
Mixed Selection 3 - Solutions

1. **Semicircle stack**
The shaded area may be divided into a 2×1 rectangle plus a semicircle plus two quarter circles (all of radius 1). Hence the total area is that of the rectangle plus a circle of radius 1, Making $2 + \pi$.



2. **Four leaf clover**
The radius of each disc in the figure is equal to half of the side-length of the square, i.e. $1/\pi$.
Because the corners of a square are right-angled, the square hides exactly one quarter of each disc. So three-quarters of the perimeter of each disc lies on the perimeter of the figure. Therefore, the length of the perimeter is $4 \times \frac{3}{4} \times 2\pi \times \frac{1}{\pi} = 6$.

3. **Pentagonal area**
The area of the pentagon is the area of a rectangle of length b and breadth a plus that of a triangle of base b and height $(c - a)$.

Hence the area is $1/2b(a + c)$.

4. **Semicircular design**
The shaded area can be seen to be the similar to the overall shape. However, it has radii that are half those of the corresponding larger shape. This means that the area is multiplied by a factor of $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk)

Stage 4 ★★
Mixed Selection 1 - Solutions

1. Circled corners
The three angles of the triangle add to $180°$, so the combined area of the three sectors of the circles that are inside the triangle add up to half a circle with area:

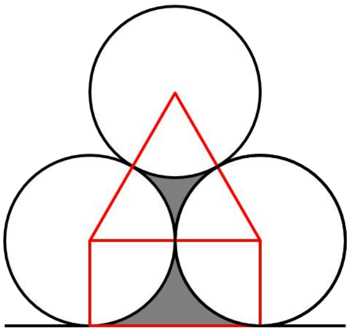
$\frac{1}{2} \times \pi \times 2^2 = \frac{4\pi}{2} = 2\pi$.

So the grey area is $(80 - 2\pi)cm^2$.

2. Rolling inside
The circumference of the circle is 2π. This is the distance its centre moves each time the circle rolls for one revolution. When the circle moves from one corner to an adjacent corner, its centre moves a distance 2, so the circle makes $1/\pi$ revolutions. As it needs to do this four times before the circle returns to its original position, the number of revolutions is $4/\pi$.

3. Sticky tape
The cross section of the $20m$ tape has area $\pi(4^2 - 3^2)cm^2 = 7\pi\ cm^2$. Therefore, the $80m$ tape should have a cross-section area $28\pi\ cm^2$. Hence, the outer radius of the $80m$ roll will be approximately $\sqrt{37}cm$.

4. Wood pile perimeter
The centres of the three circles form an equilateral triangle, so each of the sectors marked is $1/6$ of a circle. The shape below is a rectangle, so the sectors are $1/4$ of a circle. The curved portion of the perimeters is therefore the same as each of the circles: $2\pi \times 5 = 10\pi cm$.



The straight part is the same length as two radii, so is $10cm$ long.

The perimeter of the shaded shapes is therefore $(10 + 10\pi)cm$.

These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk)

Stage 4 ★★★
Mixed Selection 1 - Solutions

1. In or out

Let the radii be r. Enclose the circle in a square with sides $2r$.

The unshaded area of the square consists of 4 quadrants (quarters of circles) of radius r.

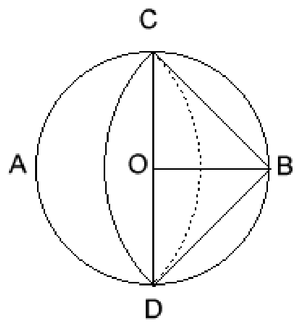
The total area of the square is $4r^2$ and the area of each quadrant is $\frac{\pi r^2}{4}$.

So the shaded area is $4r^2 - \pi r^2 = (4-\pi)r^2$. Therefore, the fraction of the circle that is shaded is $\frac{(4-\pi)r^2}{\pi r^2} = \frac{4}{\pi} - 1$.

2. Snake eyes

Let O be the centre of the circle and let the points where the arcs meet be C and D respectively. $ABCD$ is a square since its sides are all equal to the radius of the arc CD and $\angle ACB = 90°$ (angle in a semicircle).

In triangle OCB, $CB^2 = OC^2 + OB^2$; hence $CB = \sqrt{2}$ cm. The area of the segment bounded by arc CD and diameter CD is equal to the area of the triangle BCD, i.e.
$\left(\frac{1}{4}\pi\left(\sqrt{2}\right)^2 - \frac{1}{2}\times\sqrt{2}\times\sqrt{2}\right) = \left(\frac{\pi}{2} - 1\right) cm^2$.

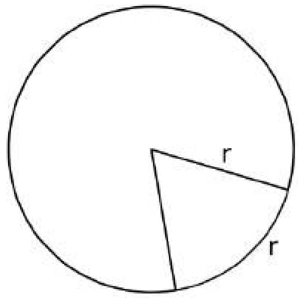


The unshaded area in the original figure is therefore $(\pi - 2)\ cm^2$. Now the area of the circle is $\pi\ cm^2$, and hence the shaded area is $2\ cm^2$.

A fuller solution is available at: https://nrich.maths.org/12997/solution

3. Clown hats

Let the radius of the circular piece of cardboard be r. The diagram shows a sector of the circle which would make one hat, with the minor arc shown becoming the circumference of the base of the hat.



The circumference of the circle is $2\pi r$. Since $6r < 2\pi r < 7r$. we can cut out 6 hats in this fashion.

Moreover, the area of cardboard unused in cutting out *any* 6 hats is less than the area of a single hat. Hence there is no possibility that more than 6 hats could be cut out.

4. Roll on

In returning to its original position, the centre of the disc moves around the circumference of a circle of diameter $3d$, i.e. a distance $3\pi d$. As the disc does not slip, its centre moves a distance πd when the disc makes one complete turn about its centre, so 3 complete turns are made.