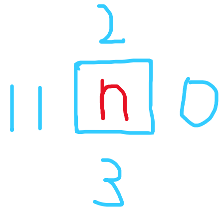
NRICH Connecting Averages Submission:

Today, I'll be presenting my various methods to solving each type of grid. Some are simple and only require basic calculations. While others may be more complicated and require lots of algebra. Please note that I use a few shortcuts to save time. Throughout my submission, I'll be using real examples generated using NRICH, and all formulas will be made using Latex. Let's begin.

1 * 1 Grid:

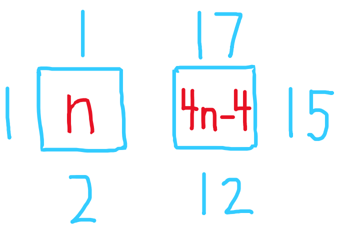


This is the simplest example. Let's set the number in the center as n. n in this case is the value of the sum of its neighbors divided by 4. So, we add up the numbers on the outside, then divide them by 4. In the example above, that would be:

$$\frac{11+2+0+3}{4}=\frac{16}{4}=4$$

The 1 * 1 Grid also provides a unique type of problem. Out of all the variations of the grid, this is the only one to have the inside of the grid already filled, and we are asked to find the number on the outside. This is also very simple. Just multiply the number in the center by 4, subtract the other 3 numbers and you get the answer.

1 * 2 Grid:



Now we progress on to the 1 * 2 Grid. In the example above, we set the leftmost box as n. So, the box on the right must be 4n – 4. This is because the sum of the number on the 4 sides must be 4n. In this way we can set up an equation:

$$\frac{n+17+15+12}{4}=4n-4$$

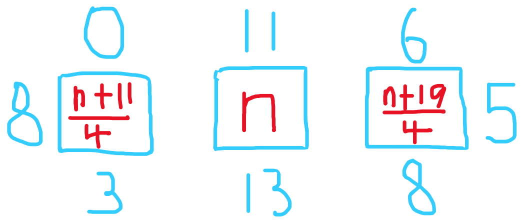
$$n+44=16n-16$$

$$15n=60$$

$$n=4$$

So, in this case the box on the left is 4, and the box on the right is 4 * 4 – 4 = 12.

1 * 3 Grid:



This question also uses a similar method as the last question. This time, I set the center box as n, and already calculated the value of the boxes next to n in terms of n. So now we can set up an equation again and solve for n:

$$\frac{\frac{n+11}{4}+\frac{n+19}{4}+11+13}{4}=n$$

$$\frac{2n+30}{4}+24=4n$$

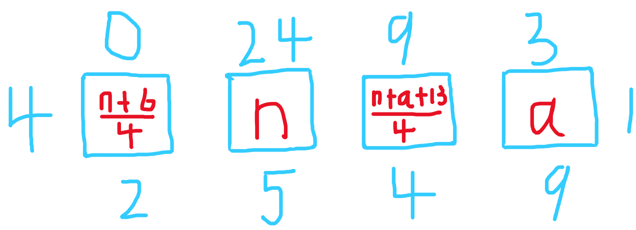
$$2n+30+96=16n$$

$$14n=126$$

$$n=9$$

So, therefore we can deduce that the value of the box on the left is 5, the box in the middle has a value of 9, and the box on the right has a value of 7.

1 * 4 Grid:



Now it starts to get difficult. We set two variables instead of one as one variable isn't enough for all 4 boxes. In this case, we can create simultaneous equations and solve for n and a.

$$\frac{\frac{n+6}{4}+\frac{n+a+13}{4}+24+5}{4}=n$$

$$\frac{2n+19+a}{4}+29=4n$$

$$2n+19+a+116=16n$$

$$a = 14n - 135$$

Now that we know the value of a in terms of n, we can create the second equation and therefore solve for n in terms of a:

$$\frac{\frac{n+a+13}{4} + 3 + 1 + 9}{4} = a$$

$$\frac{n+a+13}{4} + 13 = 4a$$

$$n + a + 13 + 52 = 16a$$

$$n = 15a - 65$$

Finally, we have our simultaneous equation:

$$a = 14n - 135$$

$$n = 15a - 65$$

Now, we can solve for a and n, we'll solve for a first:

$$a = 14(15a - 65) - 135$$

$$a = 210a - 910 - 135$$

$$209a = 1045$$

$$a = 5$$

Now that we have the value of a, we can directly substitute it into the first equation:

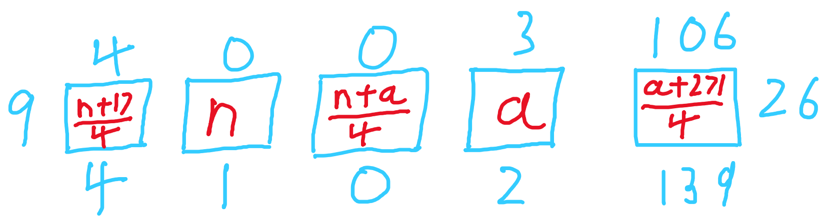
$$5 = 14n - 135$$

$$140 = 14n$$

$$n = 10$$

Now we have the values of n and a. So, the value of the boxes from left to right are 4, 10, 7 and 5, respectively.

1 * 5 Grid:



Now for the 1 * 5 Grid, we also create 2 variables, then create simultaneous equations, being a in terms of n and n in terms of a. Then we can solve for both n and a. As shown in the example above, I set the second and fourth box's value as n and a, respectively. I have already calculated the values of each box in terms of n and a, for higher efficiency.

First, we solve for a in terms of n:

$$\frac{\frac{n+17}{4}+\frac{n+a}{4}+1+0}{4}=n$$

$$\frac{n+17}{4}+\frac{n+a}{4}+1=4n$$

$$n+17+n+a+4=16n$$

$$a+21=14n$$

$$a=14n-21$$

Now we solve for n in terms of a.

$$\frac{\frac{n+a}{4}+\frac{a+271}{4}+3+2}{4}=a$$

$$\frac{n+a}{4}+\frac{a+271}{4}+5=4a$$

$$n+a+a+271+20=16a$$

$$n+291=14a$$

$$n=14a-291$$

Finally, we can substitute one of the equations in the other equation, and solve for both n and a:

$$n=14(14n-21)-291$$

$$n=196n-294-291$$

$$195n=585$$

$$n=3$$

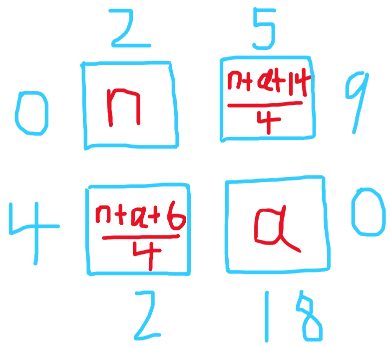
Therefore, we can substitute the value of n into the first equation:

$$a=14(3)-21$$

$$a=21$$

Hence, the values of the boxes are: 5, 3, 6, 21 and 73 respectively.

2 * 2 Grid:



Again, we set two variables, n and a. Now we're using the exact same method for the 1 * 4 and 1 * 5 Grid: Creating a simultaneous equation.

First, we solve for a in terms of n:

$$\frac{\frac{n+a+14}{4}+\frac{n+a+6}{4}+2+0}{4}=n$$

$$\frac{n+a+14}{4}+\frac{n+a+6}{4}+2=4n$$

$$n+a+14+n+a+6+8=16n$$

$$2a=14n-28$$

$$a=7n-14$$

Now we solve for n in terms of a:

$$\frac{\frac{n+a+14}{4}+\frac{n+a+6}{4}+18+0}{4}=a$$

$$\frac{n+a+14}{4}+\frac{n+a+6}{4}+18=4a$$

$$n+a+14+n+a+6+72=16a$$

$$2n=14a-92$$

$$n=7a-46$$

Now we solve for n using substitution:

$$n=7(7n-14)-46$$

$$n=49n-98-46$$

$$48n=144$$

$$n=3$$

So now we can substitute the value of n in the first equation:

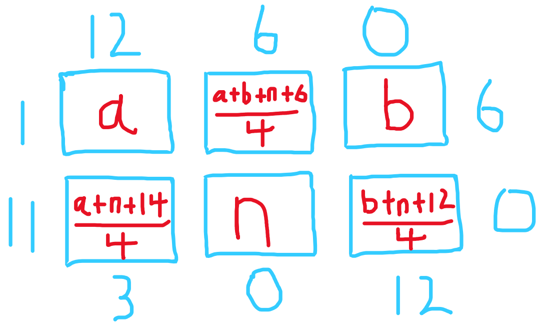
$$a=7(3)-14$$

$$a=7$$

Therefore, the values of the boxes, (from left to right, then from up to down) are 3, 6, 4 and 7.

2 * 3 Grid:

Now it starts to get difficult: we must set 3 variables instead of 2. Meaning we must create 3 equations instead of two. In the example below, I used the variables n, a, and b. And solved for the values of the boxes in terms n, a, and b.



Now let's create 3 equations, each will be one variable in terms of the others. Note that I won't write them in fraction.

First let's solve for n:

$$\frac{a+n+14}{4}+\frac{a+b+n+6}{4}+\frac{b+n+12}{4}+0=4n$$

$$a+n+14+a+b+n+6+b+n+12=16n$$

$$2a+2b+32=13n$$

$$13n=2a+2b+32$$

Next let's solve for a:

$$\frac{a+b+n+6}{4}+\frac{a+n+14}{4}+12+1=4a$$

$$a+b+n+6+a+n+14+52=16a$$

$$2n+b+72=14a$$

$$14a=2n+b+72$$

Finally, we solve for b:

$$\frac{a+b+n+6}{4}+\frac{b+n+12}{4}+6+0=4b$$

$$a+b+n+6+b+n+12+24=16b$$

$$a+2n+42=14b$$

$$14b = a + 2n + 42$$

Now, we have three equations. You can solve it however you lke, but this is my method:

- 13n = 2a + 2b + 32
- 14a = b + 2n + 72
- 14b = a + 2n + 42

we can substitute the second equation into the first by finding the LCM (Lowest Common Multiple) of the value of a (Which is 14).

So now the first two equations are:

- 91n = 14a + 14b + 224
- 14a = b + 2n + 72

Now we can substitute the second equation into the first, and end up with:

- 89n = 15b+296

We can also substitute the third equation into the first, by finding the LCM of b of the first and third equation (Which is 14). By doing that, we get:

- 91n = 14a + 14b + 224
- 14b = a + 2n + 42

Therefore:

- 89n = 15a + 266

Now we can see that 15a + 266 = 15b + 296 (because they both equal 89n), therefore we can solve for a in terms of b:

- a = b + 2.

Now let's change all the values of a to b + 2 in the first two equations, then solve for n.

- 13n = 4b + 36
- 13b = 2n + 44

We can find the LCM of the two equation's n and get:

- 26n = 8b + 72
- 169b = 26n + 572

Therefore, we can use substitution and solve for b:

$$169b = (8b + 72) + 572$$

$$161b = 644$$

$$b = 4$$

Because a = b + 2, a = 6, so we don't need a separate equation to solve for a. Now we can solve for n using the equation 26n = 8b + 72:

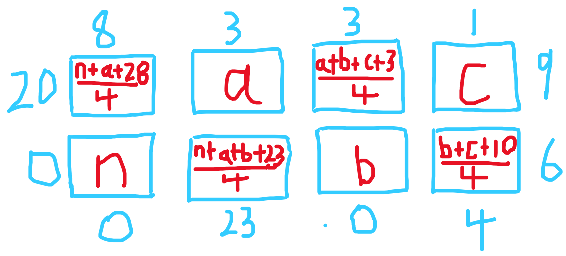
$$26n = 8(4) + 72$$

$$26n = 104$$

$$n = 4$$

Now we have all the values for n, a and b! We can finally substitute the values of n, a and b into the values we set up earlier within the boxes. So, the value of the boxes (from left to right, then from up to down) are 6, 5, 4, 6, 4 and 5.

2 * 4 Grid:



Now it gets even more difficult. We must set up 4 variables in total. This is because for every box to have a value of numbers or variables, the boxes around it must all have a number or variable value. For this Grid, I'm using the variables n, a, b and c. All the boxes have already been filled with values, see above for the example. Now we need to create 4 equations to find all 4 values.

The first equation is:

$$\frac{\frac{n+a+28}{4} + \frac{n+a+b+23}{4} + 0 + 0}{4} = n$$

$$\frac{n+a+28}{4} + \frac{n+a+b+23}{4} = 4n$$

$$n + a + 28 + n + a + b + 23 = 16n$$

$$2a + b + 51 = 14n$$

$$14n = 2a + b + 51$$

The second equation is:

$$\frac{\frac{n+a+28}{4} + \frac{n+a+b+23}{4} + \frac{a+b+c+3}{4} + 3}{4} = a$$

$$\frac{n+a+28}{4} + \frac{n+a+b+23}{4} + \frac{a+b+c+3}{4} + 3 = 4a$$

$$n + a + 28 + n + a + b + 23 + a + b + c + 3 + 12 = 16a$$

$$2b + c + 2n + 66 = 13a$$

$$13a = 2b + c + 2n + 66$$

The third equation is:

$$\frac{\frac{n+a+b+23}{4} + \frac{a+b+c+3}{4} + \frac{b+c+10}{4} + 0}{4} = b$$

$$\frac{n+a+b+23}{4} + \frac{a+b+c+3}{4} + \frac{b+c+10}{4} = 4b$$

$$n + a + b + 23 + a + b + c + 3 + b + c + 10 = 16b$$

$$2a + 2c + n + 36 = 13b$$

$$13b = 2a + 2c + n + 36$$

The fourth equation is:

$$\frac{\frac{a+b+c+3}{4} + \frac{b+c+10}{4} + 1 + 9}{4} = c$$

$$\frac{a+b+c+3}{4} + \frac{b+c+10}{4} + 10 = 4c$$

$$a + b + c + 3 + b + c + 10 + 40 = 16c$$

$$a + 2b + 53 = 14c$$

$$14c = a + 2b + 53$$

So now we have 4 equations:

- 14n = 2a + b + 51
- 13a = 2b + c + 2n + 66
- 13b = 2a + 2c + n + 36
- 14c = a + 2b + 53

Again, you can manually solve it however you like, but I'm just showing one of my methods. Mine might not be the most efficient, professional or popular method; it's just a demonstration.

So, we can sum the second and third equation to get:

- 11a + 11b = 3c + 3n +102

Now, we can sum the first and fourth equation to get

- 14c + 14n = 3a + 3b + 104

We can increase both equation's values to get:

- 33a + 33b = 9c + 9n +306
- 154c + 154n = 33a + 33b + 1144

So, we can substitute the first equation into the second, and we get:

- c + n = 10

We can also increase the values of the two equations differently to get:

- 154a + 154b = 42c + 42n +1428
- 42c + 42n = 9a + 9b + 312

Now we can substitute the second equation into the first and find the sum of a + b:

- a + b = 12

Now can simplify all the equations:

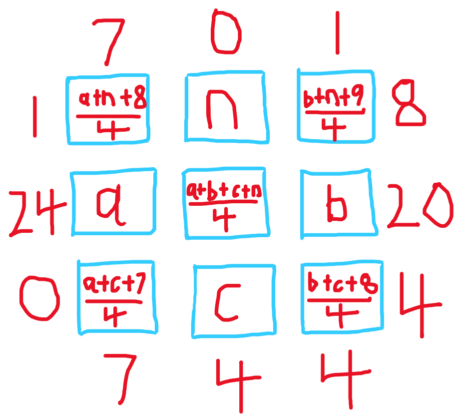
- 14n = a + 63
- 13a = 2b + n+ 76
- 13b = 2a + c + 46
- 14c = b + 65

Now I'm using a little trick. Instead of manually calculating and creating many equations. This saves much more time. Note that the first equation is 14n = a + 63. a must be 7 and n must be 5. If n was to be 6, a would have to be 21, and it cannot be because a + b = 12. Therefore, b would be -9. And there aren't any negative numbers. n could also be 4, but a would be -7. So, a = 7 and n = 5 are the only possible values.

We can use the same trick for the fourth equation, and use of estimation and common sense will give you the values b = 5 and c = 5.

Therefore, the values of the boxes from left to right, then from up to down are: 10, 7, 5, 5, 5, 10, 5, 5.

3 * 3 Grid:



This 3 * 3 Grid isn't that difficult and scary as it seems after we're done with the 2 * 4 Grid. The method is the same, and we only need to set four equations. As shown in the example above, I have still set the four variables n, a, b and c. And I've solved for the boxes around them. Now let's set up the equations.

The first equation will be to solve for n:

$$\frac{\frac{a+n+8}{4}+\frac{b+n+9}{4}+\frac{a+b+c+n}{4}+0}{4}=n$$

$$\frac{a+n+8}{4}+\frac{b+n+9}{4}+\frac{a+b+c+n}{4}=4n$$

$$a+n+8+b+n+9+a+b+c+n=16n$$

$$2a+2b+c+17=13n$$

$$13n=2a+2b+c+17$$

The second equation will be to solve for a:

$$\frac{\frac{a+n+8}{4}+\frac{a+c+7}{4}+\frac{a+b+c+n}{4}+24}{4}=a$$

$$\frac{a+n+8}{4}+\frac{a+c+7}{4}+\frac{a+b+c+n}{4}+24=4a$$

$$a+n+8+a+c+7+a+b+c+n+96=16a$$

$$b+2c+2n+111=13a$$

$$13a=b+2c+2n+111$$

The third equation will be to solve for b:

$$\frac{\frac{b+n+9}{4}+\frac{b+c+8}{4}+\frac{a+b+c+n}{4}+20}{4}=b$$

$$\frac{b+n+9}{4}+\frac{b+c+8}{4}+\frac{a+b+c+n}{4}+20=4b$$

$$b+n+9+b+c+8+a+b+c+n+80=16b$$

$$a+2c+2n+97=13b$$

$$13b=a+2c+2n+97$$

The fourth equation will be to solve for c:

$$\frac{\frac{a+c+7}{4}+\frac{b+c+8}{4}+\frac{a+b+c+n}{4}+4}{4}=c$$

$$\frac{a+c+7}{4}+\frac{b+c+8}{4}+\frac{a+b+c+n}{4}+4=4c$$

$$a+c+7+b+c+8+a+b+c+n+16=16c$$

$$2a+2b+31=13c$$

$$13c=2a+2b+31$$

Now we have all four equations set up:

- $13n = 2a + 2b + c + 17$
- $13a = b + 2c + 2n + 111$
- $13b = a + 2c + 2n + 97$
- $13c = 2a + 2b + n + 31$

Next, we can combine the first and fourth equation, and the second and third equation, then simplify the result to get our simultaneous equation:

- $3c + 3n = a + b + 12$
- $3a + 3b = c + n + 52$

Now, we can multiply the first equation by 3 to get:

- $9c + 9n = 3a + 3b + 36$
- $3a + 3b = c + n + 52$

Therefore, we can substitute the second equation into the first, then simplify to get:

- $c + n = 11$

Or we could multiply the second equation by 3 to get:

- $3c + 3n = a + b + 12$
- $9a + 9b = 3c + 3n + 156$

After substituting the first equation into the second, we get:

- $a + b = 21$

Now just like the 2 * 4 grid, we can simplify all 4 equations:

- $13n = c + 59$
- $13a = b + 133$
- $13b = a + 119$
- $13c = n + 73$

Now, we can use the same trick we used in the 2 * 4 Grid. But in this case, it is unnecessary. You can use the slightly guesswork trick we used earlier, but right now we can set up simultaneous equations easily. Let's use the first equation as an example:

- $13n = c + 59$
- $c + n = 11$

We can sum the two equations, and solve to get:

- $n = 5$ and $c = 6$.

Now we'll create another simultaneous equation using the second equation:

- $13a = b + 133$
- $a + b = 21$

Therefore, after summing both equations and solving, we get:

- $a = 11$ and $b = 10$.

With the values of a, b, c and n, we can substitute the values into the boxes and easily acquire the answers. In this case, from left to right, then from up to down, the values of the boxes respectively are: 6, 5, 6, 11, 8, 10, 6, 6 and 6.

Finally, thank you for spending time to read my proofs.