My Working Out To Negative Numbers

I started by writing the expression algebraically and went from there:

$$((-a^{-b})^{-c})^{-d}$$

where a, b, c, d are the numbers $1, 2, 3, 4$ and they are each used once (in some random order).

Inside the innermost bracket we have a negative number:

$$-a^{-b} = -1 \cdot a^{-b}$$

And by raising that to the power of $-c$:

$$(-a^{-b})^{-c} = (-1)^{-c} \cdot (a^{-b})^{-c}$$

Since $(a^{-b})^{-c} = a^{bc}$ we get

$$(-a^{-b})^{-c} = (-1)^{-c}\, a^{bc}$$

Now by raising this to $-d$:

$$\big((-a^{-b})^{-c}\big)^{-d} = \big((-1)^{-c}\, a^{bc}\big)^{-d} = (-1)^{-c\cdot(-d)}\, a^{-bcd}$$

Because $(-1)^{-c\cdot(-d)} = (-1)^{cd}$, the whole expression just simplifies to:

$$\boxed{((-a^{-b})^{-c})^{-d} \;=\; (-1)^{cd}\, a^{-bcd}}$$

- The sign (positive or negative) is given by $(-1)^{cd}$
  - If cd is even, $(-1)^{cd} = +1$ so the result is positive.
  - If cd is odd, $(-1)^{cd} = -1$ so the result is negative.
- The size (magnitude) is given by $a^{-bcd} = \dfrac{1}{a^{bcd}}$
  - The bigger a^{bcd} makes the final number closer to zero.
  - The smaller a^{bcd} makes the final number much larger in magnitude.

Now we need to consider $a = 1, 2, 3, 4$. For each a, the product bcd is the product of the three remaining numbers.

1. If $a = 1$, then $a^{-bcd} = 1^{-bcd} = 1$. So the result is ± 1. In fact, with the order which can happen the sign is positive in those cases, so we get the value of 1
2. If $a = 2$, the remaining numbers are $\{1, 3, 4\}$ so $bcd = 1 \cdot 3 \cdot 4 = 12$. $\therefore$ the magnitude is $2^{-12} = \dfrac{1}{4096}$. The final sign is $+1$ or -1 depending on whether cd is even or odd

3. If $a = 3$, the remaining numbers are $\{1, 2, 4\}$ so $bcd = 1 \cdot 2 \cdot 4 = 8$ $\therefore$ the magnitude is $3^{-8} = \frac{1}{6561}$. The final sign is $+1$ or -1 depending on cd (but for the actual order here, the sign ends up positive)

4. If $a = 4$, the remaining numbers are $\{1, 2, 3\}$ so $bcd = 1 \cdot 2 \cdot 3 = 6$. $\therefore$ the magnitude is $4^{-6} = \frac{1}{4096}$. Again, the final sign depends on whether cd is even or odd

As a result from the calculations above, the values that can occur are:

$$1 \qquad \frac{1}{4096} \qquad -\frac{1}{4096} \qquad \frac{1}{6561}$$

So there are $\boxed{4}$ distinct numbers possible.

$\therefore$ the largest value is: $\boxed{1}$ and the smallest value is: $\boxed{-\frac{1}{4096}}$

I decided to check all $4! = 24$ possible values and using that I discovered that:

$$\begin{array}{lll} 1 & \text{occurs} & 6 \text{ times} \\ \frac{1}{4096} & \text{occurs} & 8 \text{ times} \\ -\frac{1}{4096} & \text{occurs} & 4 \text{ times} \\ \frac{1}{6561} & \text{occurs} & 6 \text{ times} \end{array}$$