Finding the Lowest Common Multiple of Consecutive Integers

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On the 'nrich' website, there is a problem called "Multiple Surprises". It asks:

Can you find a few sets of ten consecutive numbers in which:

The first is a multiple of 1
The second is a multiple of 2
The third is a multiple of 3
The fourth is a multiple of 4
The fifth is a multiple of 5
The sixth is a multiple of 6
The seventh is a multiple of 7
The eighth is a multiple of 8
The ninth is a multiple of 9
The tenth is a multiple of 10?

In a generalised form, the problem looks like this:
Find an integer 'm' where:

M + 1 is a multiple of 1.
M + 2 is a multiple of 2.
M + 3 is a multiple of 3.
...
M + n is a multiple of n.

Here is a solution to the problem:

Consider integer 'k' where $k \in [1, n]$

If M + k is a multiple of k, then $M + k = a \times k$, where a is an integer.

This implies $M = (a - 1) \times k$, and that M is also a multiple of k.

Therefore, M is a multiple of all the values less than n, or M is a common multiple of all values between 1 and n. (M = CM(1 . . . n)

The smallest possible (non-trivial[1]) common multiple is the Lowest Common Multiple: LCM(1 . . . n)

The problem has now become **what is the lowest common multiple of all the integers between 1 and n?**
Once this value is found, by adding each number up to 'n', you get a sequence of consecutive integers where the u^{th} term is a multiple of u.

If M is a common multiple of 1 . . . n, then:

$\frac{M}{1}, \frac{M}{2}, \frac{M}{3}, \frac{M}{4}, \ldots$ and $\frac{M}{n}$ are all integers.

How does one test if a number is an integer?

The cosine function is a trigonometric function which oscillates between 1 and -1 infinitely. But where does it hit 1 and -1?

$cos(x) = \pm 1$, when $x = \pi \times k$, where k is an integer.

This means that:

$$cos(\pi x) = \begin{cases} \pm 1 & \text{for x is an integer} \\ (-1, 1) & \text{for x is not an integer} \end{cases}$$

We can eliminate the ± 1 by squaring all values:

$$cos^2(\pi x) = \begin{cases} 1 & \text{for x is an integer} \\ [0, 1) & \text{for x is not an integer} \end{cases}$$

This gives us a test for integers.

[1] Zero is a common multiple, and a valid solution to the problem, as any number of consecutive numbers after zero are each multiples of themselves. However, this is a trivial result and isn't particularly interesting.

Using this test to find common multiples.

We can use our new-found test to see if a number 'M' is a common multiple of 1 . . . n by testing each fraction $\frac{M}{1}$, $\frac{M}{2}$, $\frac{M}{3}$, $\frac{M}{4}$, . . . and $\frac{M}{n}$.

If each of them is an integer, the test will output 1 for each of them, thus the sum of the output must equal n:

$$\sum_{k=1}^{n} cos^2(\pi \frac{M}{k}) =$$
$$\begin{cases} n & \text{for M is a common multiple} \\ [0, n) & \text{for M is not a common multiple} \end{cases}$$

We can divide the results by n and floor the result to get a binary test for M being a common multiple:

$$\lfloor \frac{1}{n} \sum_{k=1}^{n} cos^2(\pi \frac{M}{k}) \rfloor =$$
$$\begin{cases} 1 & \text{for M is a common multiple} \\ 0 & \text{for M is not a common multiple} \end{cases}$$

This is because all the values between 0 and n are mapped to values between 0 and 1 after being divided by n, and then when they are floored, they go to zero.

Now we have a test for common multiples: how does one convert it into a formula?

Converting a test into a formula.

$\sum_{i=0}^{B} \lfloor \frac{1}{n} \sum_{k=1}^{n} cos^2(\pi \frac{i}{k}) \rfloor =$ Number of common multiples $\leq$ B

When x is an integer:

$$\lfloor \frac{1}{x} \rfloor =$$
$$\begin{cases} 1 & \text{for x = 1} \\ 0 & \text{for x > 1} \end{cases}$$

This means that $\sum_{x=0}^{\infty} \lfloor \frac{1}{f(x)} \rfloor$ counts the number of positive values of x where $f(x) = 1$.

Let $f(x) = \sum_{i=0}^{x} \lfloor \frac{1}{n} \sum_{k=1}^{n} cos^2(\pi \frac{i}{k}) \rfloor$:

$\sum_{x=0}^{\infty} \lfloor \frac{1}{f(x)} \rfloor$ now counts the number of values of x where the number of common multiples less than x = 1.

0 is a common multiple of any set of consecutive integers. Therefore, if the number of common multiples less than x is 1, then $x <$ LCM$(1 \ldots n)$.

Thus, $\sum_{x=0}^{\infty} \lfloor \frac{1}{f(x)} \rfloor$ is counting the number of values below the LCM$(1 \ldots n)$ including zero.

This sum is just the LCM$(1 \ldots n)$.

Therefore, the LCM$(1 \ldots n) = \sum_{j=0}^{\infty} \lfloor \frac{1}{\sum_{i=0}^{j} \lfloor \frac{1}{n} \sum_{k=1}^{n} cos^2(\pi \frac{i}{k}) \rfloor} \rfloor$

The ∞ bound on the $\sum$ is present because it goes through all integers to check if they are less than the LCM. We can replace this with an upper bound for the LCM$(1 \ldots n)$.

What is the upper bound for the LCM$(1 \ldots n)$?

When you take the LCM of two integers, any factors that are shared between the two numbers only have to appear in the product once.

Assuming that all the integers between 1 and n are coprime with each other, they will not share any factors. This means that every factor in all of the numbers between 1 and n must be multiplied together to make the LCM.

This is the same as multiplying each number from 1 to n together.

This means that the upper bound is n factorial (n!).

Thus, we can rewrite our formula:

$$\text{LCM}(1 \ldots n) = \sum_{j=0}^{n!} \lfloor \frac{1}{\sum_{i=0}^{j} \lfloor \frac{1}{n} \sum_{k=1}^{n} cos^2(\pi \frac{i}{k}) \rfloor} \rfloor$$

However, even this formula is quite inefficient, as it checks every number from 0 to n!. We know that LCM$(1 \ldots n)$ must be a multiple of n, so we only have to check multiples of n.

Between 0 and n!, there are $\frac{n!}{n} = (n-1)!$ multiples of n. So the limit of the first sum is $(n-1)!$. Then, within the second sum, we check $j \times n$ as this will represent each multiple of j.

Because we are increasing in gaps of n, instead of 1, we have to multiply the whole sum by n to account for the values we skipped.

The formula is now:

$$\text{LCM}(1\ldots n) = n\sum_{j=0}^{(n-1)!}\left\lfloor \frac{1}{\sum_{i=0}^{jn}\lfloor \frac{1}{n}\sum_{k=1}^{n} cos^2(\pi \frac{i}{k})\rfloor} \right\rfloor$$

For $n = 10$ this formula gives 2520.

That's the answer to the original question:

2520 + 1 = 2521 which is a multiple of 1
2520 + 2 = 2522 which is a multiple of 2
2520 + 3 = 2523 which is a multiple of 3
2520 + 4 = 2524 which is a multiple of 4
2520 + 5 = 2525 which is a multiple of 5
2520 + 6 = 2526 which is a multiple of 6
2520 + 7 = 2527 which is a multiple of 7
2520 + 8 = 2528 which is a multiple of 8
2520 + 9 = 2529 which is a multiple of 9
2520 + 10 = 2530 which is a multiple of 10

Finding other sequences with the same property.

All common multiples of 1 ... n share the above property. If you have the LCM$(1\ldots n)$, you can find any common multiple by multiplying the LCM by any integer 'k'.

Proof:

let M = LCM$(1\ldots n)$:
let k be an integer where $k \in [1, n]$:

By definition: $\frac{M}{k} = a$ where a is an integer.
It follows that: $\frac{n\times M}{k} = n \times a$ where n is another integer.

Because n and a are both integers, $n \times a$ must also be an integer. This means that $M \times n$ is a multiple of k, and thus $M \times n$ is a valid common multiple of all integers between 1 and n.

This means that, for the 'Multiple Surprises' problem, any run of 10 consecutive integers after a multiple of 2520 is a valid answer.