Question: Can you find a set of five integers that doesn't include three integers that add up to a multiple of 3?

Answer: No.

Question: If not, can you provide a convincing argument that you can always find three integers that add up to a multiple of 3?

Proof by contradiction:

All numbers divided by 3 either has a remainder of 0, 1 or 2.

$$x \bmod 3 = 0$$
$$y \bmod 3 = 1$$
$$z \bmod 3 = 2$$

If a number mod 3 = 1 and another number mod 3 = 2, then their remainders add up and their remainder becomes 0.

$$y \bmod 3 + z \bmod 3 = 0$$

So, we can simplify our question. We are trying to find 5 numbers with only 0, 1, 2 that 5 of them add up to a multiple of 3.

First, let's rule out all the sets with 3 of one number.

$$\{0,0,0, x, x\}$$
$$\{1,1,1, x, x\}$$
$$\{2,2,2, x, x\}$$

This is because when choosing three numbers, if you choose three of the same number, their sums are always a multiple of 3.

This results in the rest of the sets containing two of one number, another two of one number, and one of the last number.

$$\{x, x, y, y, z\}$$

This sort of set will always result in three distinct numbers: 0, 1 and 2.

However, with these three numbers, we can always pick them out, resulting in 0 + 1 + 2, which is a multiple of three, meaning the three numbers' sum is a multiple of three.

Overall, no matter what the integers are, there will ALWAYS be three that adds up to a multiple of three.