Pre-STEP School online lecture series: Lecture 4 – Question 1

STEP Mathematics III Summer 1997

7 For each positive integer n, let

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots ;$$
$$b_n = \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots .$$

(i) Evaluate b_n.

(ii) Show that $0 < a_n < 1/n$.

(iii) Deduce that $a_n = n!\mathrm{e} - [n!\mathrm{e}]$ (where $[x]$ is the integer part of x).

(iv) Hence show that e is irrational.

Please provide answers to the following discussion questions. Don't include full calculations in your responses, just explore the question and try to anticipate routes through it.

1. b_n is an infinite sum – the question seems to assume that it can be evaluated. Which types of infinite sum can you evaluate? Is b_n one of these types? If so, why? What standard formula will you need?

2. a_n is also an infinite sum. How can you show that an infinite sum is less than something? Why is a_n trickier to deal with than b_n? How can b_n help you with a_n?

3. Here you need to think about the infinite series definition of e. What might be the best thing to do to get an insight into what is required here?

4. Being irrational is defined in a negative way – it means you are not a rational. Does this give you a hint for a proof strategy? What other proofs of irrationality have you seen? What strategy did they use? What would happen if you started in the same way with this question? Why will it be significant that part (iii) is true for *all* n?

Submit your answers by e-mail to stepeasterschool@maths.org by Friday 23rd March 2012 with the subject line: Lecture 4 Question 1