Factorising with Multilink

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If you want to make a shape in base x with 1 x^2 piece, 7 x pieces and 12 unit pieces, you will end up with a shape that has an area of $x^2 + 7x + 12$.

$x^2 + 7x + 12 = (x + 3)(x + 4)$, which means that a shape with this area could be made by a rectangle of these dimensions. Both $x + 3$ and $x + 4$ are greater than x, which means that such a rectangle could accommodate the length of the x^2 and x pieces, and this means that such a rectangle can be made.

The 'end-on' special case occurs in the case of $x = 3$ because 12 is divisible by 3.

$3^2 + 7 \times 3 + 12 = 3 \times (3 + 7 + 4)$ (a rectangle of dimensions 3x14)

This example can help us reach a generalisation.

Generalisation

First, we must establish some facts. In the expression $x^2 + bx + c$, b and c will always be positive integers, since it would be nonsensical to build a rectangle using negatively-valued pieces or with pieces that have been cut in half. This means our rectangle side lengths will always be greater than the length of the larger pieces, so we don't need to consider whether an x^2 piece will fit in our rectangle.

From the example we can see that if the expression $x^2 + bx + c$ can be split into two factors, then a rectangle can be made from the combination of an x^2 piece, b lots of x pieces and c lots of units pieces. Also, we can see that the special cases of 'end-on' rectangles occur when c is divisible by a chosen value of x for any value of b.

Case studies:

$x^2 + 100x + m$

If m can be expressed as $m = k(100 - k)$for any integer k so that $1 < k < 100$, then a rectangle can be made with dimensions $(x + k) \times (x + 100 - k)$. This is a square in the case of $k = 50$.

Special cases arise when m is divisible by x, as mentioned previously.

$x^2 + mx + 12$

$12 = 3 \times 4 = 2 \times 6$, so if $m = 3 + 4 = 7$ or $m = 2 + 6 = 8$, then you can have rectangles with dimensions $(x + 3) \times (x + 4)$ or $(x + 2) \times (x + 6)$ respectively.

End-on rectangles can be made when x is 2, 3, 4 or 6.

$x^2 + mx + 100$

$100 = 2 \times 50 = 4 \times 25 = 5 \times 20 = 10 \times 10$, so like before, you can have dimensions of $(x + 50) \times (x + 2)$, $(x + 25) \times (x + 4)$, $(x + 5) \times (x + 20)$ or $(x + 10) \times (x + 10)$ (a square) when m takes the values 52, 29, 25 or 20 respectively.

End-on rectangles can be made when x is 2, 4, 5, 10, 20, 25 or 50.

By testing different factors, these ideas appear to extend to expressions with a coefficient of x^2 greater than 1, which you may expect since the number of x^2 pieces doesn't influence the logic I used in the example at the beginning. It would therefore appear that the only requirements on any set of x^2, x and units pieces for it to be capable of forming a rectangle is for its quadratic expression to be expressible as the product of two factors.