When circles of radii $\frac{1}{2d^2}$ and $\frac{1}{2c^2}$ are tangents to each other and to the x-axis, touching $(0,\frac{b}{d})$ and $(0,\frac{a}{c})$ respectively (Figure 1), a right angled triangle with sides $\frac{1}{2d^2}+\frac{1}{2c^2}$, $\frac{1}{2d^2}-\frac{1}{2c^2}$ and $\frac{a}{c}-\frac{b}{d}$ can be constructed (Figure 2).
By Pythagoras' theorem,

$$(\frac{1}{2d^2}+\frac{1}{2c^2})^2=(\frac{1}{2d^2}-\frac{1}{2c^2})^2+(\frac{a}{c}-\frac{b}{d})^2$$
$$(\frac{ad-bc}{cd})^2=(\frac{c^2+d^2}{2c^2d^2})^2-(\frac{c^2-d^2}{2c^2d^2})^2$$
$$\frac{(ad-bc)^2}{c^2d^2}=\frac{4c^2d^2}{4c^4d^4}$$
$$(ad-bc)^2=\frac{4c^4d^4}{4c^4d^4}$$
$$(ad-bc)^2=1$$
$$|ad-bc|=1$$

For the two circles to be tangents to each other, $\frac{b}{d}$ and $\frac{a}{c}$ must be Farey neighbours.

Drawing a third circle touching the x-axis at $(0,\frac{a+b}{c+d})$ (Figure 3), it can now be shown to be a tangent to both the circles.

With x the distance between the centres of the circle on the left and the middle circle (Figure 4), a right angled triangle can be formed with sides x, $(\frac{a+b}{c+d}-\frac{b}{d})$, $(\frac{1}{2d^2}-\frac{1}{2(c+d)^2})$.
By Pythagoras' theorem,

$$x^2=(\frac{a+b}{c+d}-\frac{b}{d})^2+(\frac{1}{2d^2}-\frac{1}{2(c+d)^2})^2$$
$$x^2=\frac{(ad-bc)^2}{d^2(c+d)^2}+\frac{c^2(c+2d)^2}{4d^4(c+d)^4}$$

Recalling that $|ac-bd|=1$,

$$x^2=\frac{1}{d^2(c+d)^2}+\frac{c^2(c+2d)^2}{4d^4(c+d)^4}$$
$$x^2=\frac{c^4+4c^3d+8c^2d^2+8cd^3+4d^4}{4d^4(c+d)^4}$$

For the circles to be touching, the distance between the centres must be the sum of the radii.

$$x=\frac{1}{2d^2}+\frac{1}{2(c+d)^2}$$
$$x=\frac{c^2+2cd+2d^2}{2d^2(c+d)^2}$$
$$x^2=\frac{c^4+4c^3d+8c^2d^2+8cd^3+4d^4}{4d^4(c+d)^4}$$

The two values of x correspond, therefore the two circles touch. In the same way it can be shown that the right circle and middle circle touch. Therefore the new circle added is also in a Farey sequence.

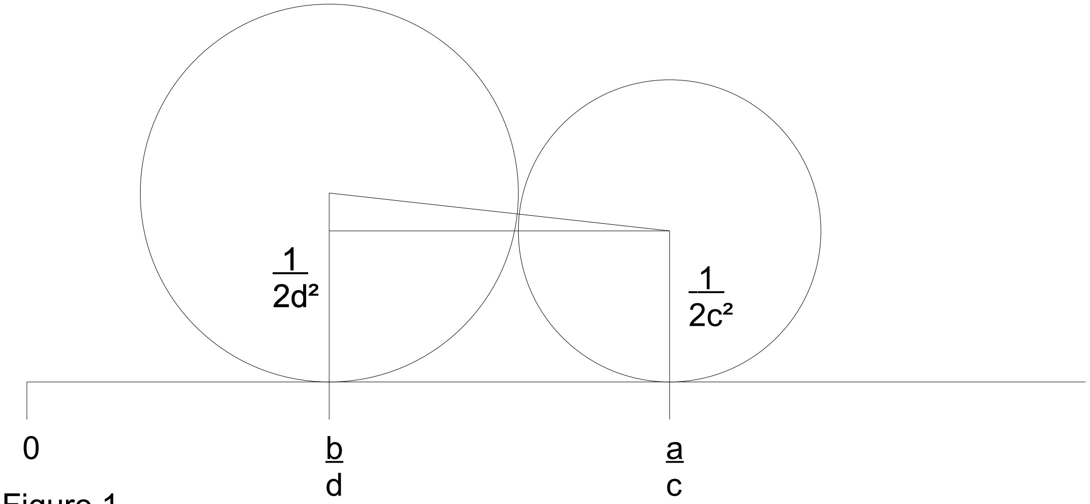

Figure 1

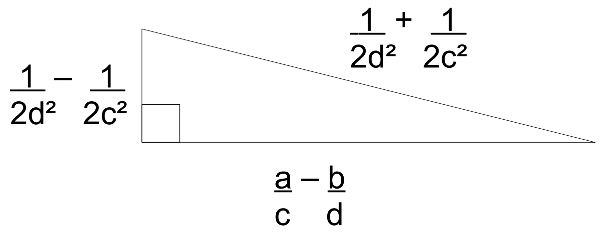

Figure 2

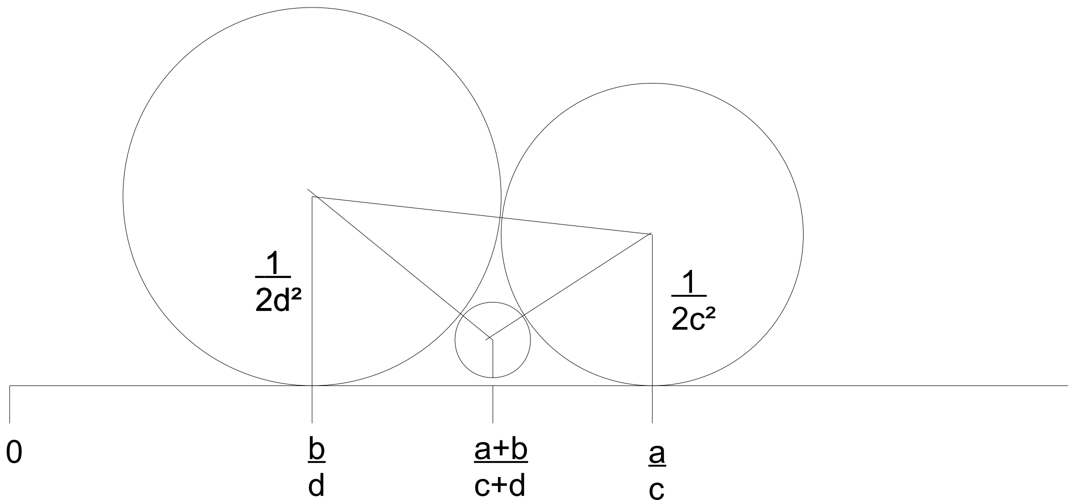

Figure 3

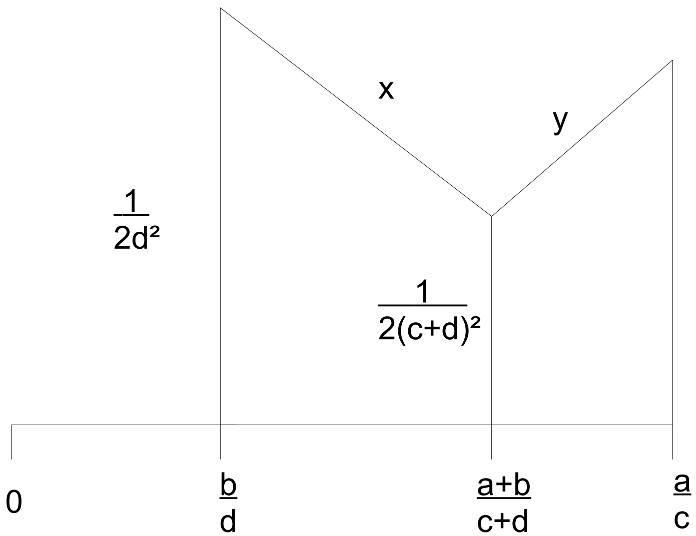

Figure 4