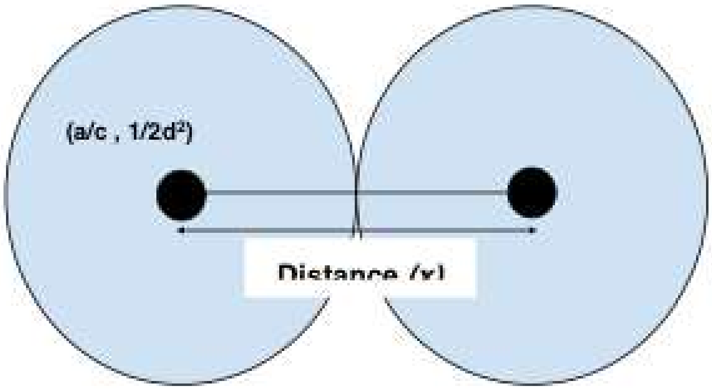
Vignesh Balaji
Year 12 Hymers College

Can you prove that for any touching circles in the interactivity above, $|ad - bc| = 1$?



In order for two circles to intersect at **only one point** the sum of the radii must equal the distance (x).

$\therefore$ $x = 1/2d^2 + 1/2c^2$ (since the radii of the circles are $1/2d^2$ and $1/2c^2$ respectively)

Then using pythagoras' theorem we can deduce that:

$x = \sqrt{(a/c - b/d)^2 + (1/2d^2 - 1/2c^2)^2}$

This is because the hypotenuse is the sum of the change in x values squared and the change in y values squared. So now if we substitute into our original equation:

$\sqrt{(a/c - b/d)^2 + (1/2d^2 - 1/2c^2)^2} = 1/2d^2 + 1/2c^2$

We square both sides and rearrange:

$(a/c - b/d)^2 + (1/2d^2 - 1/2c^2)^2 = (1/2d^2 + 1/2c^2)^2$

$(a/c - b/d)^2 = (1/2d^2 + 1/2c^2)^2 - (1/2d^2 - 1/2c^2)^2$

Now let us focus on the RHS:

Let $(1/2d^2 + 1/2c^2)^2 - (1/2d^2 - 1/2c^2)^2 = (x + y)^2 - (x - y)^2$

$(x + y)^2 - (x - y)^2 \equiv x^2 + 2xy + y^2 - x^2 + 2xy - y^2$

$\equiv 4xy$

$\equiv 4(1/4c^2d^2)$

$\equiv 1/c^2d^2$

Substitute this back into the main equation:

$(a/c - b/d)^2 = 1/c^2d^2$

If we rearrange the LHS we are left with this:

$(ad - bc/cd)^2 = 1/c^2d^2$

$(ad - bc)^2/c^2d^2 = 1/c^2d^2$

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$(ad - bc)^2 \quad = c^2d^2/c^2d^2$ (Multiply both sides by c^2d^2)
$(ad - bc)^2 \quad = 1$
$\therefore\ ad - bc \quad = 1$ (square root both sides)

This is true for all values of a,b,c **and** d.

Can you prove that, given two such circles which touch the *x* axis at *b/d* and *a/c*, the circle with centre (*a*+*b*/*c*+*d*,1/2(*c*+*d*)2) and radius 1/2(*c*+*d*)2 is tangent to both circles?

To prove this we can essentially use the same principle as we did before; if this circle was tangent to both circles then it will touch **each circle** at only **one point** on **both circumferences.**

Let "$c + d = x$" (to make the calculations easier to read):

$\therefore\ (a/c - (a + b)/x)^2 + (1/2c^2 - 1/2x^2)^2 = (1/2c^2 + 1/2x^2)^2$

Notice that the same equation distance^2 = change in x squared add change in y squared. We can then rearrange:

$(a/c - (a + b)/x)^2 = (1/2c^2 + 1/2x^2)^2 - (1/2c^2 - 1/2x^2)^2$

The same pattern has come up on the RHS, so:

$(a/c - (a + b)/x)^2 = 4(1/4c^2x^2)$
$(a/c - (a + b)/x)^2 = 1/c^2x^2$

Rearrange the LHS:

$(ax - ac - bc)^2/c^2x^2 = 1/c^2x^2$

Note $x = c + d$, this can now be substituted into the numerator on the LHS:

$(ac + ad - ac - bc)^2/c^2x^2 = 1/c^2x^2$
$(ad - bc)^2/cx = 1/c^2x^2$

We now know that $ad - bc = 1$, and so we are left with:

$1/c^2x^2 = 1/c^2x^2$

This is clearly true for all values of c and x, thus proving that this circle is tangent to the other circle. If we repeat this process, except for the x,y and r values of the circle it comes as the same answer except the c replaced with a d, exemplifying that the circle is tangent to both the other circles.