Not so simple pendulum 2

(NOTE from Ed: Sorry for the typo! Its corrected in the problem) From the description given I'm guessing that the differential equation is meant to be:

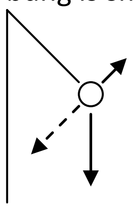
$$m\frac{d^2x}{dt^2} + k^2x = 0$$

Because x is the angle and not y. Note that x is measured in radians. If you've done the not so simple pendulum 1 problem then you will notice that this is the same equation and therefore m is the mass of the bung in kg, but in this equation k^2/m = g/L where g is the gravitational field strength and L is the length of the string the bung is attached to, and therefore k^2 is measured in kg s^{-2}. Also as proved before the general solution is:

$$x = x_0 \cos\left(\frac{tk}{\sqrt{m}}\right)$$

Where x_0 is the initial angle in radians.

The modelling assumption that there is a force acting against the direction of motion that is proportional to the velocity would exist in the real world, but this would be air resistance and not friction. If it were friction then you would have to deal with moments. For the given differential equation, the units of λ times s^{-1} (because d/dt divides the units by time which are seconds in SI units) must equal the units of k^2, and k^2 is measured in kg s^{-2} so λ must be measured in kg s^{-1}. The assumption that λ is constant is just as good an assumption that g stays constant as the bung changes height, in other words it does change a little because the air is slightly thinner as the bung gets higher, but it changes so little that it really doesn't make a big difference to the end result. Now rather than solving the given differential equation I will derive and solve my own (hopefully) more accurate equation using the assumption that the bung is small enough to obey Stokes law.



On the diagram the force pointing down is gravity, the dotted line is the resolved force of gravity, and the other line is drag/air resistance. We can see the resolved force of gravity is:

$$F_g = mg\sin(x)$$

$$F_g \approx mgx$$

Where m is the mass of the bung, g is gravity and x is the angle between the vertical and the string and is very small. The force is negative because the direction it is acting is against the direction of motion. Now using Stokes law the force from the air resistance is:

$$F_d = 6\pi rnv$$

Where r is the radius of the bung, n is the coefficient of viscosity for the fluid the bung is in, and v is the velocity of the bung. Now if we consider direction in these equations we can work out the resultant force and turn this into a differential equation:

$$F_r = F_g - F_d$$
$$F_r = -mgx - 6\pi rnv$$
$$ma = -mgx - 6\pi rnv$$
$$m\frac{d^2s}{dt^2} + 6\pi rn\frac{ds}{dt} + mgx = 0$$
$$mL\frac{d^2x}{dt^2} + 6\pi rnL\frac{dx}{dt} + mgx = 0$$
$$\frac{d^2x}{dt^2} + \frac{6\pi rn}{m}\frac{dx}{dt} + \frac{g}{L}x = 0$$
This can be written in a simpler form:
$$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + \mu x = 0,$$
$$\lambda = \frac{6\pi rn}{m}, \mu = \frac{g}{L}$$
Now for the fun bit – solving it. First let's make the assumption that the solution is in the form of an exponential and try and solve it.
$$x = Ae^{rt}$$
$$Ar^2x + A\lambda rx + A\mu x = 0$$
$$r^2 + \lambda r + \mu = 0$$
$$r = \frac{-\lambda \pm \sqrt{\lambda^2 - 4\mu}}{2}$$
Now µ will always have to be greater than λ in order for any resultant force to occur, so we can assume that r will be in the form of a complex number. If it isn't a complex number then it will be an exponential graph which never touches 0 and therefore the viscosity of the fluid or the gravitational field are way too far off their actual value.
$$r = p \pm qi,$$
$$p = \frac{-\lambda}{2}, q = \sqrt{\left(\frac{\lambda}{2}\right)^2 - \mu}$$
$$\therefore x = Ae^{pt \pm qit}$$
$$x = Ae^{pt}e^{\pm qit}$$
$$x = e^{pt}(A\cos(qt) + B\sin(qt))$$
As you will notice this is similar to what we are asked to differentiate in the problem. I will have to differentiate it anyway so we can find the value of A and B:

$$x_0 = e^0(A\cos(0) + B\sin(0))$$
$$x_0 = 1(1A + 0B)$$
$$\therefore A = x_0$$
$$\omega = \frac{dx}{dt}$$
$$\omega = \frac{d}{dt}\left(e^{pt}\right)\left(A\cos(qt) + B\sin(qt)\right) + e^{pt}\frac{d}{dt}\left(A\cos(qt) + B\sin(qt)\right)$$
$$\omega = pe^{pt}(A\cos(qt) + B\sin(qt)) + e^{pt}\left(-Aq\sin(qt) + Bq\cos(qt)\right)$$
$$\omega_0 = pe^0(A\cos(0) + B\sin(0)) + e^0\left(-Aq\sin(0) + Bq\cos(0)\right)$$
$$\omega_0 = Ap + Bq$$
$$B = \frac{\omega_0 - px_0}{q}$$

Now we have A and B we can write the solution as:

$$x = \frac{x_0\cos\left(t\sqrt{\left(\frac{\lambda}{2}\right)^2 - \mu}\right) + \frac{2\omega_0 + \lambda x_0}{\sqrt{\lambda^2 - 4\mu}}\sin\left(t\sqrt{\left(\frac{\lambda}{2}\right)^2 - \mu}\right)}{e^{\frac{\lambda t}{2}}}$$

As I said before, this way would be slightly more accurate because we do every step and understand what each constant means, but this seems to mean that the solution will look a lot more complicated. Here is a plot of the solution on geogebra: