<u>All Tangled Up</u>

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It is harder to find out the sequence using turns and twists to reach any fraction than disentangle it. Now we will propose a 'reverse sequence' method – we find a sequence using turns and "backward-twists" to get a fraction to 0, then we reverse the sequence and replace those "backward-twists" with the normal twists (or "forward-twists") to get the required sequence to reach the fraction.

We define a 'backward-twist' as $x \rightarrow x-1$, which is an inverse operation of a 'forward-twist'. A turn is an inverse operation of itself.

All simplified fractions can be split into the following categories:

1. -1/n
2. -m/n
3. n/m
4. 1/n
5. –n/m
6. m/n

where $1< m<n$

Next, we will prove that Case 1 & 2 fractions can be taken back to 0 using turns and 'backward-twists'. The other cases can be reduced to Case 2. The resulting 'backward' sequence is the reverse order of the required 'tangle-up' sequence when all 'backward-twists" are replaced by "forward-twists".

<u>Case 1: -1/n</u>

We turn $-1/n$ once then backward-twist n times. This is the reverse order of the sequence produced when we forward-twist n times from 0, then turn once.

<u>Case 2: -m/n</u>

We turn $-m/n$ once into n/m then continue 'backward-twisting' until we get the first negative fraction.

Let $n=km+r$,

$n/m= (km+r)/m=k+ r/m$, so we need 'backward-twist' k+1 times to get the first negative fraction:

$r/m-1=- (m-r)/m$.

As $m<n$ and $(m-r)<m$, both numerator and denominator of the original fraction $-m/n$

have been reduced. We keep repeating the above procedure to -(m-r)/m until the fraction becomes of the form -1/a (a is a positive integer). This is Case 1. Then the required 'tangle-up sequence' is the reverse order of the sequence obtained above.

e.g. –m/n= -4/9
$\therefore$k=2, r=1

Then the 'backward sequence' generated by the reverse sequence method is

$-4/9 \rightarrow 9/4 \rightarrow 5/4 \rightarrow 1/4 \rightarrow -3/4 \rightarrow 4/3 \rightarrow 1/3 \rightarrow -2/3 \rightarrow 3/2 \rightarrow 1/2 \rightarrow -1/2 \rightarrow 2 \rightarrow 1 \rightarrow 0$

Then the 'Tangle-up sequence' is

$0 \rightarrow 1 \rightarrow 2 \rightarrow -1/2 \rightarrow 1/2 \rightarrow 3/2 \rightarrow -2/3 \rightarrow 1/3 \rightarrow 4/3 \rightarrow -3/4 \rightarrow 1/4 \rightarrow 5/4 \rightarrow 9/4 \rightarrow -4/9$

Case 3: n/m

With k+1 backward-twists, we get $n/m -(k+1) = -(m-r)/m$, which is Case 2.

Case 4: 1/n

With one backward-twist, $1/n$ becomes – $(n-1)/n$, which is Case 2.

Case 5: -n/m

With one turn and one backward-twist, -n/m becomes $m/n -1= -(n-m)/n$. As $n>m$ so $-(n-m)/n$ is Case 2.

Case 6: m/n

With one backward-twist, m/n becomes $-(n-m)/n$, which is Case 2.

Furthermore, a negative integer n can be turned to give $1/n$, which is Case 4, and positive integer n can be backward-twisted n times to get 0.

Therefore, any rational number can be turned and back-twisted to 0. When reversing the order, we can use turns and twists to reach it. A simple rule to achieve these is given below.

General Rule:

Starting with the targeted fraction/number

If a positive number/fraction is reached, backward-twist once; if it is a negative number/fraction, turn once.

When zero is reached, reverse the order.

Q.E.D