Tangles

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1 How it all started

This is a personal story of discovery that began in February 2007 when a colleague of mine, Ian Short, showed me an interesting trick he'd seen in a lecture given by the mathematician John Conway in Cambridge in 1998. At the very start of that lecture, he said this:

> What I like doing is taking something that other people thought was complicated and difficult to understand, and finding a simple idea so that any fool – and in this case, *you* – can understand the complicated thing.

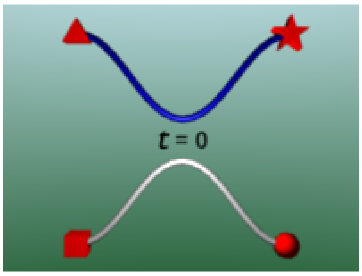
He's telling us there are lots of really difficult ideas out there that really aren't that difficult at all – if only we can see them from a better angle. Well, after reading the lecture transcript it still took quite a long time to find the better angle to see his 'little trick', but he had captured my interest and I ended up having a lot of fun investigating it, and learnt quite a lot of mathematics on the way.

2 Conway's rope trick

An edited transcript of the lecture was available on the web at the time I wrote this article [2, page 10] , but It's best to do the trick for real rather than work from diagrams. If you can't find the lecture on the web, try this book [9] instead. There is also a streamed video transcript of a similar lecture delivered by Conway at http://www.msri.org/publications/ln/msri/1997/ldt/c

You need two ropes – skipping ropes are good – and four volunteers. Each volunteer keeps hold of one skipping rope handle.

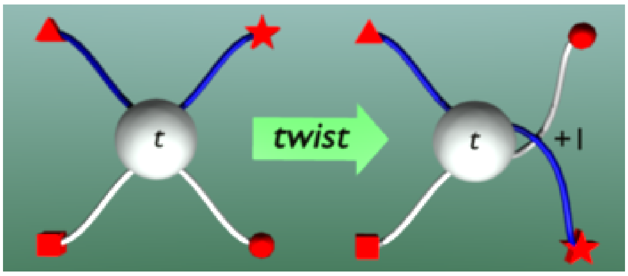
Now, arrange the 4 volunteers so the skipping ropes look like this from above:



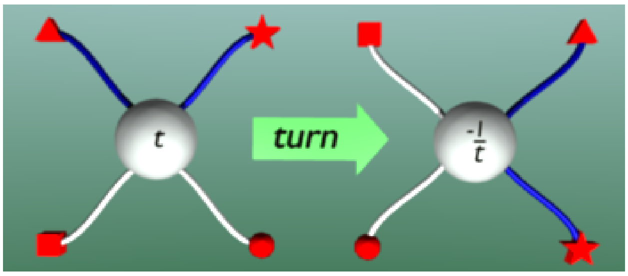
This is the first and simplest example of a *tangle*. It's so simple that you might even dispute that it is a tangle, but instead of just saying that the ropes are untangled I'm going to say that this is a tangle of value zero, or $t = 0$.

Now, our volunteers are allowed to make just two moves. Conway calls these moves "Twist 'em up" and "Turn 'em round" and makes a kind of a square dance out of them. We'll just call the moves *twist* and *turn* for short.

In *twist*, the two volunteers on the right swap places – it's important that the volunteer in the top right corner always transfers the rope over the rope from the bottom right corner. This adds a twist to the tangle so its value is now one more than it was before.



In *turn*, all four volunteers turn clockwise through a quarter turn. This has a strange effect on the value of the tangle - you have to divide it into 1 and change its sign. A value of $\frac{2}{3}$ will become $-\frac{3}{2}$; a value of $3 = \frac{3}{1}$ will become $-\frac{1}{3}$; and as a final example, a value of $-\frac{5}{8}$ will become $\frac{8}{5}$. Oh, and I mustn't forget to say that *turn* takes 0 to ∞ and ∞ to 0.



3 Doing the trick

Now, beginning with the zero tangle we can start the square dance!

Start	Move	End
0	*twist*	1
1	*twist*	2
2	*twist*	3
3	*turn*	$-\frac{1}{3}$
$-\frac{1}{3}$	*twist*	$\frac{2}{3}$
$\frac{2}{3}$	*twist*	$\frac{5}{3}$
$\frac{5}{3}$	*twist*	$\frac{8}{3}$
$\frac{8}{3}$	*turn*	$-\frac{3}{8}$
$-\frac{3}{8}$	*twist*	$\frac{5}{8}$
$\frac{5}{8}$	*twist*	$\frac{13}{8}$

That's probably far enough for now. You should have something that is nicely twisted and tangled in the centre. Hopefully your antics have attracted an audience so you can now ask them to direct the dance, choosing either *twist* or *turn* with the aim of getting the tangle value back to zero.

When they succeed in getting the tangle value back to zero, you should find that the ropes are also magically untangled. Hopefully your audience will discover the simple rule that will help them decide whether a twist or a turn is needed at each stage of disentangling.

You might like to watch this video of the trick being enacted by the NRICH Morris Men.

4 Did I miss something?

It may seem unsurprising that the ropes untangle at the end. After all, why shouldn't they? We did keep careful track of the tangle values all the way through and we did start with an untangled value of zero. So if we're back at zero why wouldn't the ropes be untangled? Did I miss something?

Let's think about it for a minute though. It didn't seem to matter what route we took through all the possible tangles before returning to zero. Somehow, the tangle value completely captured the state of the tangle and told us all we need to know to untangle it. Are tangles really that organized? It certainly doesn't feel like that in normal life. There's either some magic or some mathematics at work here. I must admit I have a strong preference for there being mathematical explanations, and so it's time to start asking questions.

Question 1: *Exactly what sort of tangles are we able to make with Conway's rope trick?*

It's worth noticing that our square dance is not capable of producing just any old tangle. Many tangles – anything that contains reef knots or granny knots or those awkward tangles where one of the ropes is just knotted – are simply impossible to make this way. Our rope trick works with just those tangles that can be made with the square dance - no more and no less. Conway calls this subset of all possible tangles the *Rational Tangles* because his trick demonstrates that they are uniquely identified by a rational number. A rational number, by the way, is just another way of saying 'fraction' – i.e. an integer divided by another integer. Just as there are tangles that are not rational, there are also numbers that are not rational, such as $\sqrt{2}$ and π.

Question 2: *Given any rational tangle, can you always get back to the zero tangle using just* twist *and* turn *operations?*
Think about this. The answer is *Yes!*. You should be able to develop an algorithm that you can prove will always work.

Question 3: *Given any rational number, can you always find a sequence of* twist *and* turn *operations that will generate a tangle with that value?*
Again the answer is *Yes!*, and you should be able to find a method that will always work.

The fact that Conway's tangles only generate rational numbers leads to another interesting line of thought:

Question 4: *Can we give any meaning to the idea of an irrational tangle?*

If you generate a sequence of rational numbers, there is no guarantee that they will approach a rational limit. Some sequences, such as $0.9, 0.99, 0.999, 0.9999, ...$ do. Many do not. For example, the sequence $0, 0.1, 0.101, 0.101001, 0.1010010001, ...$ does not approach a rational number even though it is clearly approaching a unique finite limit.

One way to investigate this question is to look at sequences of repeated tangle moves, attempting to find tangles with rational values that approach an irrational limit. The limiting tangle – whatever that is – should in some sense be an *irrational tangle.*

For example, lets see what happens with the sequence that repeats T = *3 twists followed a turn* indefinitely. It generates the values

$$-\frac{1}{3}, -\frac{1}{3-\frac{1}{3}}, -\frac{1}{3-\frac{1}{3-\frac{1}{3}}}, \ldots$$

.

These calculations involve things called *continued fractions.* A little internet research, or indeed, this article on NRICH by Alan and Toni Beardon will quickly throw up some interesting continued fraction sequences for irrational numbers such as $\sqrt{2}$.

Oh, and you'll find a short section of drainpipe very useful. It's all explained in this video.

Here's another question:

Question 5: *Are all tangles really the same after only two* turn *moves?*

Take a rational tangle and *turn* twice. it will have turned through 180 degrees. If it's a sufficiently complicated tangle it will not look like the original tangle. However, by pulling on pairs of ropes carefully you should be able to convince yourself that indeed it is the same as the original tangle.

This bothered me a lot when I first saw the trick because that *turn* move only turns through 90 degrees and so it should take 4 turns to return the tangle to its original state. However Conway tells us that the corresponding transform on the tangle value t is

$$t \to -\frac{1}{t}.$$

That's strange, because this returns to the value t after just 2 applications.

$$\text{For example:} \quad \frac{2}{3} \to -\frac{3}{2} \to \frac{2}{3} \quad \text{in just 2 steps}$$

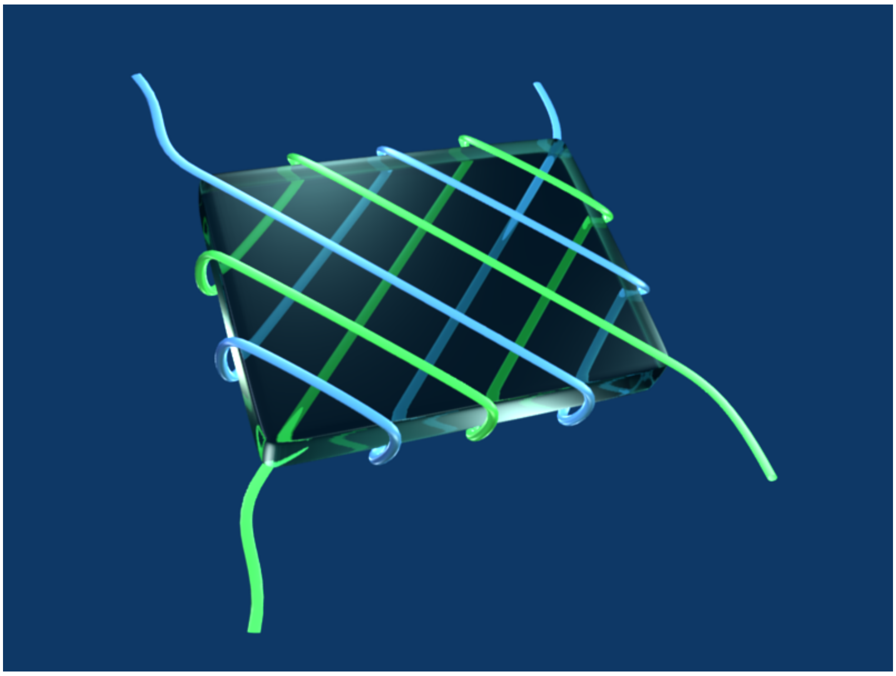
It seems that all tangles have a rotational symmetry of order 2. Now why is that?

In fact, rational tangles have even more symmetries. We've spotted that a 180 degree turn about an axis perpendicular to the rope plane leaves the tangles unchanged. This rotation swaps opposite pairs of rope handles. In our diagrams above, (star, circle)→(square,triangle). There are two other axes – the vertical and horizontal axes in the diagram – which swap (star,triangle)→circle,square), and (star,circle)→(triangle,square).

Make some complicated tangle up, and check for yourself that all these rotations leave the tangle unchanged. Think about how you could prove this and then read [4, Symmetric Tangles by Robert Crowston].

Question 6: *If tangles are so symmetrical, is there a way to make them look symmetrical?*

The awkward thing about tangles, or indeed knot theory in general, is that it can be very hard to see that two identical tangles (or knots) really are the same, because simple rope manipulations can change their appearance so easily. But maybe there is a way of arranging every tangle so it is possible to see the underlying symmetry. Well, it turns out that such a symmetrical arrangement is always possible. This is an image of the tangle 4/3 in its symmetrical form. The symmetrical form offers many insights into how the Conway Rope Trick works.



Perhaps after all this, you may be wondering why tangles are worthy of study? That leads to the next question:

Question 7: *What are tangles good for?*

Rational tangles provide the only known example in the large theory of knots, of a *complete invariant*. Much of knot theory is about telling the difference between one knot and another, and an invariant is something you can use to describe a given knot that will not change as you manipulate the knot in space. The trouble is that for knots, no invariant is powerful enough to distinguish any knot from any other knot. The tangle invariant is a useful lever to use when tackling that bigger problem. Conway first developed the rational tangles in order to help tabulate all the prime knots and links up to 11 and 10 crossings.

In [8] Joanna Lewis's article on Drawing Doodles and Naming Knots you'll be introduced to some of the earlier schemes that were used for this purpose. These early techniques were not very good at distinguishing one knot from another.

Question 8: *Where now?*

Find some ropes and start your own investigation! There's plenty to discover, and plenty more to read about. I recommend [1, The Knot Book] by Colin Adams as an excellent introduction.

References

[1] Colin C. Adams. *The Knot Book: an elementary introduction to the mathematical theory of knots*. American Mathematical Society, 2004.

[2] John Conway. The power of mathematics, 1998. URL `ftp://www.inference.phy.cam.ac.uk/pub/mackay/conway.pdf`.

[3] Peter Cromwell. *Knots and Links*. Cambridge University Press, 2004.

[4] Robert Crowston. Symmetric tangles. URL `http://nrich.maths.org/public/viewer.php?obj_id=5746`.

[5] Jay R. Goldman and Louis H. Kauffman. Knots, tangles, and electrical networks. *Advances in Applied Mathematics*, 14:267–306, 1993.

[6] Louis H. Kauffman. *Formal Knot Theory*. The Princeton Mathematical Notes. Princeton University Press, 1983.

[7] Louis H. Kauffman and Sofia Lambropoulou. Classifying and applying rational knots and rational tangles. In DeTurck, editor, *Contemporary Mathematics*, volume 304, pages 223–259, 2001.

[8] Joanna Lewis. Drawing doodles and naming knots. URL `http://nrich.maths.org/public/viewer.php?obj_id=5787`.

[9] David MacKay, editor. *Power*. Cambridge University Press, 2004.