Colour building

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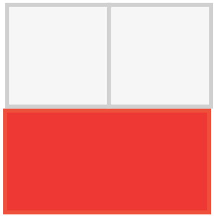
Monday, 27 January 2020

To start off, let us define a rod with its unit. We know White = 1 and Red = 2. Similarly, light green = 3 and Pink = 4

Let T_n define the number of ways of making a rod of length 'n' just by using red and white rods. We already know the following

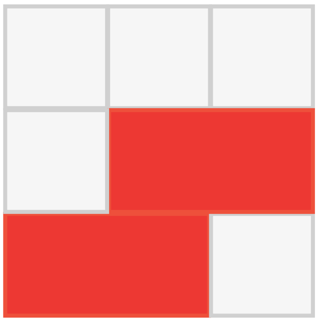
$T_1 = 1$

$T_2 = 2$

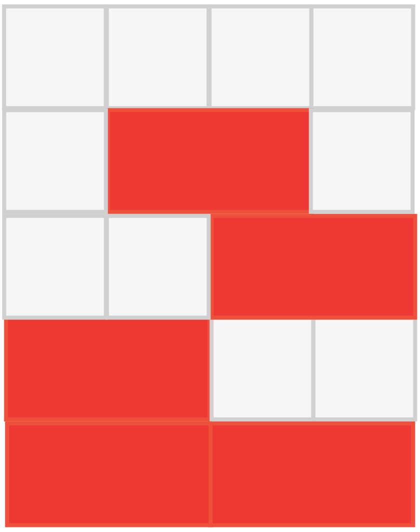


Let us look at T_3 and T_4 as well

$T_3 = 3$



$T_4 = 5$



We can count all of the possibilities systematically making sure none is left. We observe that we start to make the 'n' length rod using one white at the left most place and cover the 'n-1' rods using red and whites. The ways to cover 'n-1' length rod as defined in the beginning is $T_{n-1.}$ Next we introduce a red at the left most side and try to cover the rest of the 'n-2' remaining units with the red and the whites. The total ways of making a rod of length 'n-2' is as defined in the beginning $T_{n-2.}$ If we go further to $T_{n-3,}$ we can note that all the cases involved for this will already be in T_{n-1}, because the left most place is white in both T_{n-1} and T_{n-3}

Thus we get the explicit formula for combining a rod of length 'n' using red and white as:

$$\boxed{T_n = T_{n-1} + T_{n-2}}$$

Let us verify this for $T_{4,}$

$$T_4 = T_3 + T_2$$
$$5 = 3 + 2$$
$$5 \equiv 5$$

Using the formula we can calculate T_5, $T_{6,}$ $T_{7,}$ .....

$$T_5 = T_4 + T_3 = 5 + 3 = \underline{8}$$
$$T_6 = T_5 + T_4 = 8 + 5 = \underline{13}$$
$$T_7 = T_6 + T_5 = 13 + 8 = \underline{21}$$
$$T_8 = T_7 + T_6 = 21 + 13 = \underline{34}$$
$$T_9 = T_8 + T_7 = 34 + 21 = \underline{55}$$
$$T_{10} = T_9 + T_8 = 55 + 34 = \underline{89}$$

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$$T_n = T_{n-1} + T_{n-2}$$

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By interesting observation, you can see that that $T_n = F_{n-1}$ where F_n is the nth Fibonacci number

The task of Finding T_n requires finding the values of T_{n-1} and T_{n-2}, which require other previously known T_k.

By recurrence relation, and generating functions, we can derive a formula of T_n with just n as the variable

$T_n = T_{n-1} + T_{n-2}$

$$r^n = r^{n-1} + r^{n-2}$$

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

$$T_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$T_1 = A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

$$T_2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = 2$$

Solving For A and B further, we get

$$A = \left(\frac{5+\sqrt{5}}{10}\right) \qquad B = \left(\frac{5-\sqrt{5}}{10}\right)$$

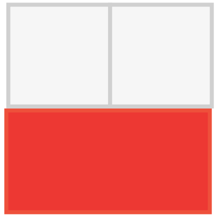
$$\text{Thus } T_n = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Where T_n denotes the number of ways to tile a Cuisenaire rod of length n using White (1) and Red (2)

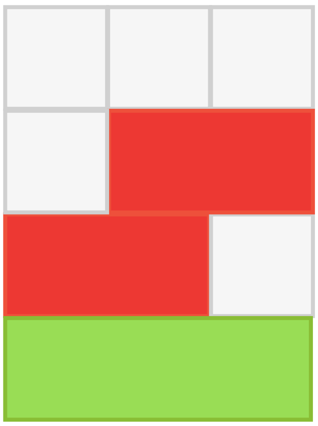
For covering a rod of length 'n' we define the number of ways to cover it using White (1), Red (2) and Green (3) by T_n^3

$T_1^3 = 1$

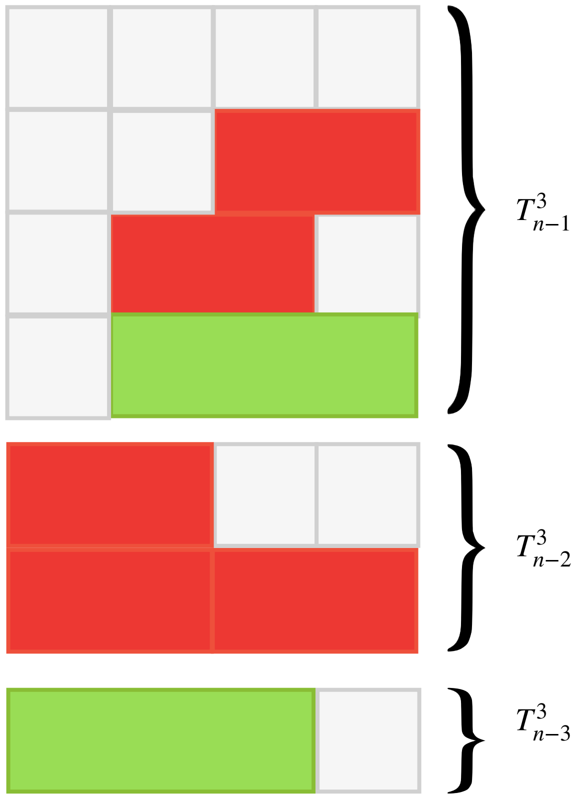
$T_2^3 = 2$



$T_3^3 = 4$



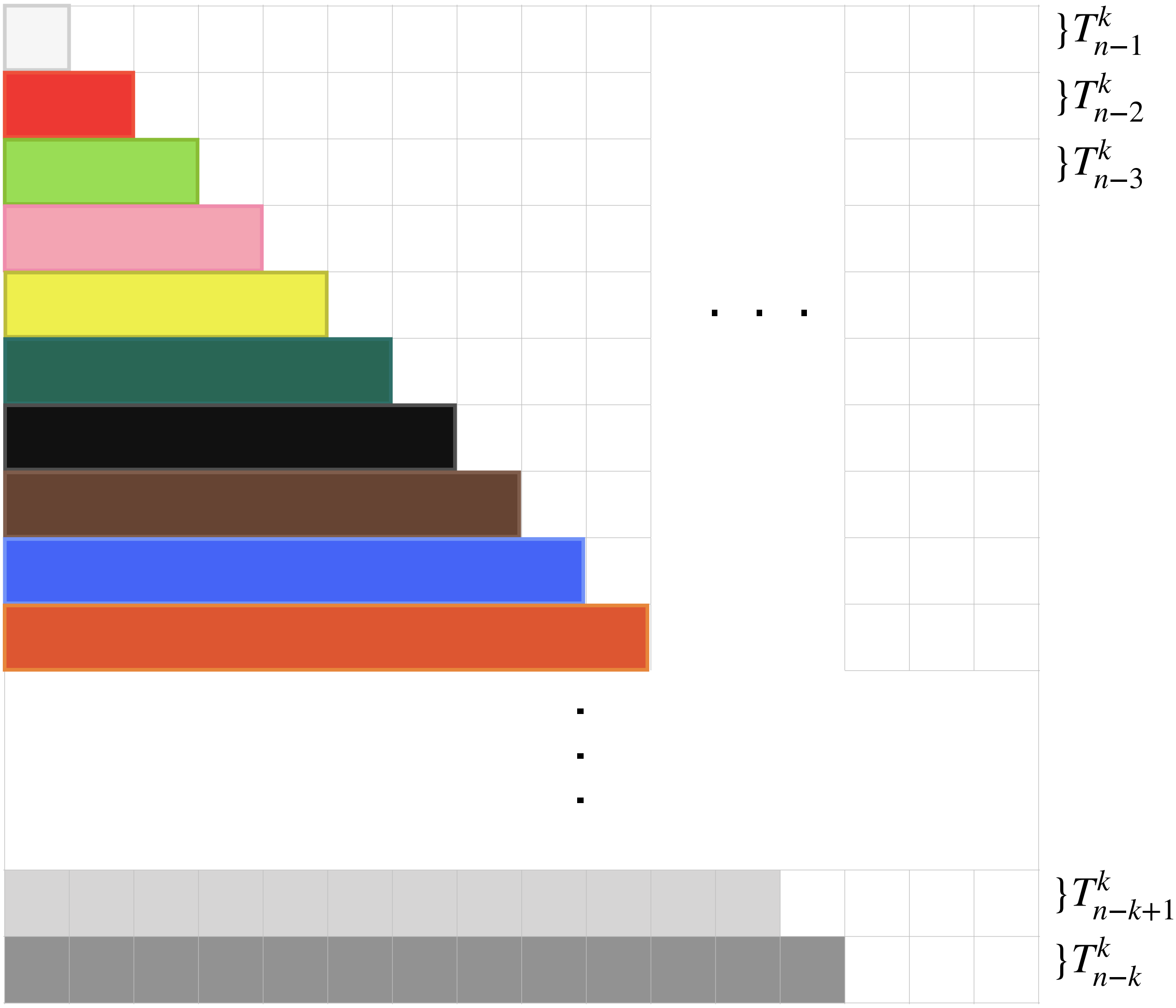
$T_4^3 = 7$



Similarly T_n^3 is the addition of T_{n-1}^3, T_{n-2}^3, T_{n-3}^3. We can see this because, When a white rod of unit 1 is placed at the left most place, the ways of covering the rest of the 'n-1' units is T_{n-1}^3. Similarly when a red rod is placed in the left most side, the ways of covering the rest of the 'n-2' units is T_{n-2}^3. And the same, for when a green rod is placed at the left most place we get T_{n-3}^3 ways to cover the 'n-3' left places.

$$\boxed{T_n^3 = T_{n-1}^3 + T_{n-2}^3 + T_{n-3}^3}$$

To cover a rod of length 'n' using all the rods till length 'k' (i.e. cover using $1,2,3\ldots k-1,k$), we define the ways to cover it by T_n^k



$$\boxed{T_n^k = T_{n-1}^k + T_{n-2}^k + T_{n-3}^k + \ldots + T_{n-k+1}^k + T_{n-k}^k}$$

So to cover a Cuisenaire rod of length 5 using All of White (1), Red (2), Green (3) and Pink (4)

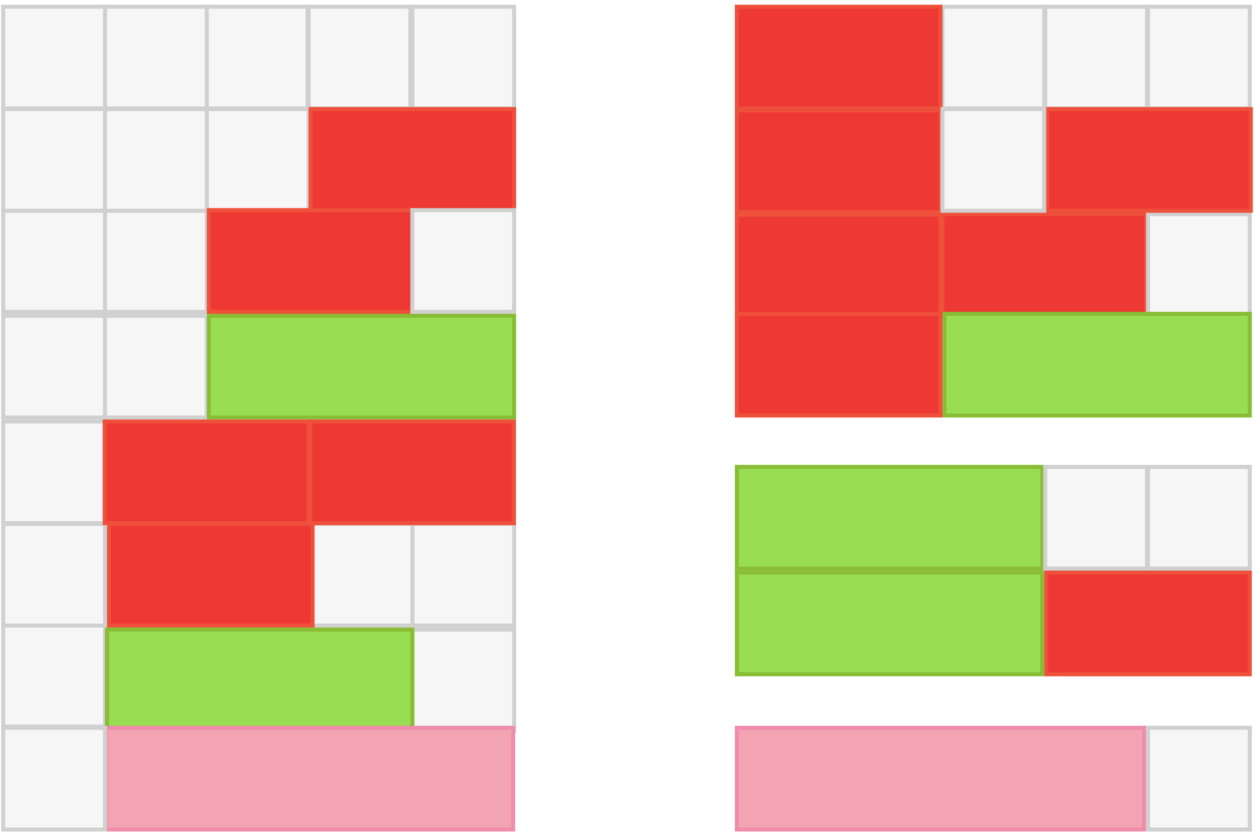
By our defined notation this should equal

$$T_5^4 = T_4^4 + T_3^4 + T_2^4 + T_1^4$$

$$T_4^4 = 8, \quad T_3^4 = 4, \quad T_2^4 = 2, \quad T_1^4 = 1$$

$$T_5^4 = 8 + 4 + 2 + 1$$

$$T_5^4 = 15$$



$$T_5^4 \equiv 15$$

We can see that our formula for finding the ways of combing a rod of length 'n' using 'k' consecutive numbers is indeed correct

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