1 Introduction

I'm going to assume that you followed the theoretical parts of the notes. In other words, if we want to maximize or minimize

$$J[y] = \int_a^b F(x, y, y')\,\mathrm{d}x$$

then we have to solve the differential equation given by the Euler-Lagrange equations (E-L equations)

$$\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}$$

2 Euclidean Geodesic

For the Euclidean Geodesic problem, we want to minimize

$$\int_a^b \sqrt{1 + y'^2}\,\mathrm{d}x$$

So, if we define

$$F(x, y, y') = \sqrt{1 + y'^2}$$

then we want to minimize

$$J[y] = \int_a^b F(x, y, y')\,\mathrm{d}x$$

as above. So we require

$$\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'}$$

(the E-L equation). For our F, we have $\frac{\partial F}{\partial y} = 0$ (because there is no y term in our F). Also, we have

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

This may be a little more difficult to appreciate (it confused the hell out of me when I first saw it), but just imagine that y' is just another variable z, and you get $F(x, y, z) = \sqrt{1 + z^2}$ which would be perfectly OK (this all makes sense if you are using partial derivatives). So, the E-L equation for this example is

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\frac{y'}{\sqrt{1 + y'^2}}$$

In other words

$$\frac{y'}{\sqrt{1 + y'^2}} = k$$

for some constant k. Squaring the equation and multiplying by $1 + y'^2$ gives you

$$y'^2 = k^2(1 + y'^2)$$

which can be simplified to give you

$$y' = \sqrt{\frac{k^2}{1-k^2}}$$

In other words $y' = c$ for some constant c. As you know, this is a straight line, so the Geodesic in the Euclidean plane is a straight line.