The problem is to maximize

$$\int_{-1}^{1} y \, \mathrm{d}x$$

subject to

$$\int_{-1}^{1} \sqrt{1+y'^2} \, \mathrm{d}x = L$$

where $L = \pi$. We choose this value of L so that this part of the problem works out nicely. You can do with more generality by integrating between $-R$ and $+R$ with a fixed length L (not necessarily πR^2), but you'll find that the value of R which maximizes the area will be $R = \sqrt{\frac{L}{\pi}}$, and this will be just a maximization problem in one variable, R, which you know how to do.

To solve this problem we need *Lagrange Multipliers* which I explained in my initial introduction to the problem. On the assumption you followed that, let's continue:

We set $F(x, y, y') = y$ and $G(x, y, y') = \sqrt{1+y'^2}$. We want to maximize

$$\int_{-1}^{1} F - \lambda G \, \mathrm{d}x$$

Plugging this into the Euler-Lagrange equation we get

$$\frac{\partial}{\partial y}(F + \lambda G) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}(F + \lambda G)$$

We have $\frac{\partial F}{\partial y} = 1$, $\frac{\partial G}{\partial y} = 0$, $\frac{\partial F}{\partial y'} = 0$ and $\frac{\partial G}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$. Plugging this into the above equation we get

$$1 = \frac{\mathrm{d}}{\mathrm{d}x}(\lambda \frac{y'}{\sqrt{1+y'^2}})$$

Integrating both sides we get

$$x + C = \frac{y'}{\sqrt{1+y'^2}}$$

Multiplying and squaring we get

$$\lambda^2 {y'}^2 = (1 + {y'}^2)(x + C)^2$$

Rearranging for y' we get

$$y' = \frac{x + C}{\sqrt{\lambda^2 - (x + C)^2}}$$

We can integrate this (by substituting $z = x + C$ and recognizing the answer) to get

$$y = -\sqrt{\lambda^2 - z^2} + D = -\sqrt{\lambda^2 - (x + C)^2} + D$$

We want to fix $y(-1) = y(1) = 0$, so we get the following two equations:

$$-\sqrt{\lambda^2 - (C-1)^2} + D = 0 \ \ (1)$$
$$-\sqrt{\lambda^2 - (C+1)^2} + D = 0 \ \ (2)$$

Subtract these two equations, square them and subtract λ^2 from the result to get $(C-1)^2 = (C+1)^2$. In other words, $C = 0$. Substituting this C back into equation (1) we get $D = \sqrt{\lambda^2 - 1}$. In other words our solution is

$$y = \sqrt{\lambda^2 - 1} - \sqrt{\lambda^2 - x^2}$$

At this point we use our integral constraint, i.e.

$$\int_{-1}^{1} G(x, y, y')\, \mathrm{d}x = \pi$$

If we work through the algebra, we get

$$G(x, y, y') = \sqrt{1 + y'^2} = \frac{\lambda}{\sqrt{\lambda^2 - x^2}}$$

Integrating this from -1 to 1 we get the equation

$$\lambda \sin^{-1} \frac{x}{\lambda} - \lambda \sin^{-1} \frac{-x}{\lambda} = \pi$$

The solution of this is given by $\lambda = 1$. Therefore we end up with the solution

$$y = -\sqrt{1 - x^2}$$

The reason we don't have

$$y = \sqrt{1 - x^2}$$

as you'd expect is that we are not maximizing the area, we are extremizing it, this means we can find a maximum or a minimum, in this case we have a minimum. At one point in the working, if you took a negative square root instead of a positive square root, you would have ended up with the maximum instead of the minimum, but it's an unimportant difference.