Pair Products

Year 8 Set 1: British International School Phuket

This is a summary of the work our students did on this task:

4 consecutive numbers

$$x, x+1, x+2, x+3$$

Product of first and last:

$$x(x+3) = x^2 + 3x$$

Product of middle two:

$$(x+1)(x+2) = x^2 + 3x + 2$$

Difference of two results:

$$x^2 + 3x + 2 - (x^2 + 3x) = 2$$

5 consecutive numbers

$$x, x+1, x+2, x+3, x+4$$

No 2 middle numbers so square the middle number

Product of first and last:

$$x(x+4) = x^2 + 4x$$

Product of middle squared:

$$(x+2)^2 = x^2 + 4x + 4$$

Difference of two results:

$$x^2 + 4x + 4 - (x^2 + 4x) = 4$$

Following the same method we get the following table:

Consecutive numbers	Difference
3	1
4	2
5	4
6	6
7	9
8	12
9	16
10	20
11	25

Students then noticed that there were separate pattens for odds and even consecutive numbers.

For the odd consecutive numbers we have:

$$1,4,9,16,25$$

This has nth term n^2 when we take 3 consecutive numbers as $n = 1$ etc.

For the even consecutive numbers we have:

$$2,6,12,20$$

This has nth term $n^2 + n$ when we take 4 consecutive numbers as $n = 1$ etc.

Prem also then extended the table back to 1 consecutive number, and then found a recursive relationship:

Consecutive terms	1	2	3	4	5	6	7	8
difference	0	0	1	2	4	6	9	12

"The difference in n consecutive numbers is the previous difference plus $\frac{n-1}{2}$ (rounded down if a decimal)." We can write this as:

$$D(n) = D(n-1) + \left\lfloor \frac{n-1}{2} \right\rfloor$$

Pratik found the fourth difference of this sequence:

Consecutive terms	1	2	3	4	5	6	7	8
difference	0	0	1	2	4	6	9	12

Which he noticed alternated between ± 2.

Even and odd consecutive numbers

Students then looked at consecutive even numbers:

4 consecutive even numbers

$$2x, 2x+2, 2x+4, 2x+6$$

Product of first and last:

$$2x(2x+6) = 4x^2 + 12x$$

Product of middle two:

$$(2x+2)(\,2x+4) = 4x^2 + 12x + 8$$

Difference of two results:

$$4x^2 + 12x + 8 - (4x^2 + 12x) = 8$$

Chloe that made the following table to summarise this data:

Consecutive even numbers	Difference
4	8
5	16
6	24
7	36
8	48
9	64
10	80
11	100

She noticed that there were 2 differences of 8 followed by 2 differences of 12 followed by 2 differences of 16. Using this she predicted that the next 2 differences would be 20. She then calculated this and found that this was correct.

Pratik then used the same method for odd numbers

$$2x+1, 2x+3, 2x+5, 2x+7$$

Which gave a difference of 8.

General case

Venya tackled the general case of *n* consecutive numbers with difference *q:*

$$x, x+q, x+2q, \ldots . x+q(n-1)$$

This has first term multiplied by last term as:

$$x\big(x+q(n-1)\big) = x^2 + xq(n-1)$$

If there are an even number of terms then the middle 2 multiplied together gives:

$$\left(x+\left(\frac{n}{2}\right)q\right)\left(x+\left(\frac{n}{2}-1\right)q\right)$$

$$= x^2 + xq(n-1) + q^2\frac{n}{2}\left(\frac{n}{2}-1\right)$$

Therefore the difference he found was:

$$q^2\frac{n}{2}\left(\frac{n}{2}-1\right)$$

For an odd number of *n* consecutive terms this method gave the middle number squared as:

$$\left(x+\left(\frac{n-1}{2}\right)q\right)^2$$

$$= x^2 + xq(n-1) + q^2\frac{n-1}{4}$$

Therefore the difference he found was:

$$q^2\frac{(n-1)^2}{4}$$

Final result:

An even number n of consecutive numbers with difference q has Pair Products given by:

$$q^2 \frac{n}{2}\left(\frac{n}{2} - 1\right)$$

An odd number n of consecutive numbers with difference q has Pair Products given by:

$$q^2 \frac{(n-1)^2}{4}$$