

1. I am sure there's perhaps a faster way of doing this but this is the solution I found. Firstly, we can attempt a little prime factorisation to notice that  $24 \mid a \iff 2^3, 3 \mid a$ . So, we can develop a two pronged attack onto this proof - one to prove that our expression is divisible by 8 and one to show it is divisible by 9.

Factorising  $n^5 - n^3$  we get  $n^3(n+1)(n-1)$ . Slipping in an  $n$  we get  $n^2((n-1)(n)(n+1))$ . That inside bracket is the product of three consecutive numbers which is always divisible by three. This can be done quite easily if you think about it, a multiple of 3 occurs every 3 numbers thus over the span of 3 consecutive numbers at least one of them is a multiple of 3 thus the product of those consecutive numbers must be divisible by 3.

To prove everything was divisible by 8, I took quite the boring method. Case one is that  $n$  was even  $\Rightarrow n = 2q, q \in \mathbb{N} \Rightarrow (2q)^3((4q^2) - 1) = 8q^3(\dots)$  thus it holds for even values. If  $n$  is odd  $\Rightarrow n = 2q+1 \Rightarrow (2q+1)^3(2q+2)(2q) = 8(4q^5 + 10q^4 + 9q^3 + 28q^2 + 4q) + 28q^2 + 4q = 8(r) + 4(7q^2 + 1)$ ,  $r \in \mathbb{N}$ . We use a very similar idea to prove that  $4(7q^2 + 1)$  is always divisible by 8 which is simple as all we need to really prove is  $(7q^2 + 1)$  is always even and then factor out a 2 to get an 8. If  $q$  is even we get odd + 1 = even and if  $q$  is odd we get the same result.

Thus  $24 \mid n^5 - n^3, \forall n \in \mathbb{N}$

2. This proof can be done quite nicely via induction ( a rather lovely tool ). First, we assume a base case - for this question, our base case is  $n = 1$  since our statement holds for all positive integers. We can then attempt to prove our base case which is rather simple for this question.  $2^{2(1)} - 1 = 3$ .

Now, we take a rather bold move where we assume that  $3 \mid 2^{2k} - 1$  for some  $k \in \mathbb{N} \Rightarrow 2^{2k} - 1 = 3q$  for some  $q \in \mathbb{N}$ . The notation may look intimidating at first if you haven't seen it before but all it says that the  $2^{2k} - 1$  is divisible by 3 which means it can be written as  $3q$  where  $q$  is an integer.

This may seem pointless at first but it will be helpful when analyse the next case for  $n = k + 1$ ,  $2^{2(k+1)} - 1$ . We can use the law of indices to help re write our expression as  $2^2 \cdot 2^k - 1$ . Now, we simply factor out a 4 but naturally you may be scared to do so because we can't easily pull out a 4 from that  $-1$  can we ? Well say we did anyways, and we had  $4(2^k - 1)$  and we expanded it all out, we'd get  $4 \cdot 2^k - 4$  which is almost what we had before ! Ah-ha, you might notice all we have to do is add three and we get back what we had. So, for  $n = k + 1$  we get,  $4(2^k - 1) + 3$

This is where the magic happens. Remember that almost random step we did in the 2nd paragrah - well look inside that bracket, it's our case for  $n = k$  so we can rewrite it again as  $4(3q) + 3$  which is simply  $3(4q + 1)$ .

This is when we have to use some logic, if we proved the statement to be true for  $n = 1$  and we assumed it to be true for  $n = k$  and using that result we proved that it's true for  $n = k + 1$  then it must be true  $\forall n \in \mathbb{N}$

3. Firstly, we can notice that if  $3 \mid n - 1 \Rightarrow n - 1 = 3q, q \in \mathbb{N}$ . Thus,  $n = 3q + 1$ . If we attempt to directly plug this into our  $n^3 - 1$  We get  $(3q + 1)^3 - 1$  which if we expand it all out gives us  $27q^3 + 27q^2 + 9q + 1 - 1$  then  $9(3q^3 + 3q^2 + 1)$  which is clearly divisible by 9.