Tri-Angled Trigonometry (NRICH Solution)

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1 Abstract

This document presents a solution to the ”Tri-Angled Trigonometry” problem published by the University of Cambridge’s NRICH website. Methods used throughout this proof include common trigonometric identities (e.g., sum and difference formula, co-function identities) alongside basic algebraic manipulation. I hope my explanations will help budding mathematicians better understand the intricacies behind solving this problem!

2 Solution (Part I)

For the first part of the problem, our aim is to show that,

$$sin^2\theta + sin^2\phi + sin^2\psi + 2sin(\theta)sin(\phi)sin(\psi) = 1 \tag{1}$$

when $\theta + \phi + \psi = \frac{1}{2}\pi$. So, factoring out a common expression $sin(\theta)$ from LHS of (1) simplifies the equation to,

$$sin(\theta)[sin(\theta) + 2sin(\phi)sin(\psi)] + sin^2(\phi) + sin^2(\psi) \tag{2}$$

To further simplify $sin(\theta) + 2sin(\phi)sin(\psi)$, let us rearrange the second original equation’s terms,

$$\theta + \phi + \psi = \frac{1}{2}\pi \tag{3}$$

$$\Rightarrow \phi + \psi = \frac{1}{2}\pi \ - \theta \tag{4}$$

And using the co-function identity $cos(\frac{1}{2}\pi \ - x) = sin(x)$ for $x = \theta$,

$$cos(\frac{1}{2}\pi \ - \theta) = sin(\theta) \tag{5}$$

$$\Rightarrow cos(\phi + \psi) = sin(\theta) \tag{6}$$

Therefore, substituting (6) into (2) gives us,

$$sin(\theta)[cos(\phi + \psi) + 2sin(\phi)sin(\psi)] + sin^2(\phi) + sin^2(\psi) \tag{7}$$

Using the sum and difference formula $cos(\phi+\psi) = cos(\phi)cos(\psi) - sin(\phi)sin(\psi)$, (7) can be expressed as,

$$sin(\theta)[cos(\phi)cos(\psi) - sin(\phi)sin(\psi) + 2sin(\phi)sin(\psi)] + sin^2(\phi) + sin^2(\psi) \quad (8)$$

$$\Rightarrow sin(\theta)[cos(\phi)cos(\psi) + sin(\phi)sin(\psi)] + sin^2(\phi) + sin^2(\psi) \quad (9)$$

Since $sin(\theta) = cos(\phi + \psi) = cos(\phi)cos(\psi) - sin(\phi)sin(\psi)$, substituting (6) into (9) gives us,

$$cos(\phi + \psi)[cos(\phi)cos(\psi) + sin(\phi)sin(\psi)] + sin^2(\phi) + sin^2(\psi) \quad (10)$$

$$\Rightarrow [cos(\phi)cos(\psi) - sin(\phi)sin(\psi)][cos(\phi)cos(\psi) + sin(\phi)sin(\psi)] + sin^2(\phi) + sin^2(\psi) \quad (11)$$

Using the difference of squares formula $(a + b)(a - b) = a^2 - b^2$ for $a = cos(\phi)cos(\psi)$ and $b = sin(\phi)sin(\psi)$, (11) can be simplified to,

$$cos^2(\phi)cos^2(\psi) - sin^2(\phi)sin^2(\psi) + sin^2(\phi) + sin^2(\psi) \quad (12)$$

Since $cos^2(\theta) = 1 - sin^2(\theta)$ (derived from the Pythagorean identity $sin^2(\theta) + cos^2(\theta) = 1$), (12) can also be expressed as,

$$[1 - sin^2(\phi)][1 - sin^2(\psi)] - sin^2(\phi)sin^2(\psi) + sin^2(\phi) + sin^2(\psi) \quad (13)$$

Distributing the terms of $[1 - sin^2(\phi)][1 - sin^2(\psi)]$ in (13) and expanding the brackets gives us,

$$1[1 - sin^2(\psi)] - sin^2(\phi)[1 - sin^2(\psi)] - sin^2(\phi)sin^2(\psi) + sin^2(\phi) + sin^2(\psi) \quad (14)$$

$$\Rightarrow 1 - sin^2(\psi) - sin^2(\phi) + sin^2(\phi)sin^2(\psi) - sin^2(\phi)sin^2(\psi) + sin^2(\phi) + sin^2(\psi) \quad (15)$$

Since like terms with opposite signs cancel each other out, we are left with,

$$1 - \cancel{sin^2(\psi)} - \cancel{sin^2(\phi)} + \cancel{sin^2(\phi)sin^2(\psi)} - \cancel{sin^2(\phi)sin^2(\psi)} + \cancel{sin^2(\phi)} + \cancel{sin^2(\psi)} \quad (16)$$

$$\Rightarrow 1 \quad (17)$$

This proves that LHS = RHS of (1). (Q.E.D.)

3 Solution (Part II)

For the second part of the problem, our aim is to show that $x = sin(\frac{1}{10}\pi)$ satisfies,

$$8x^3 + 8x^2 - 1 = 0 \quad (18)$$

when $sin^2\theta + sin^2\phi + sin^2\psi + 2sin(\theta)sin(\phi)sin(\psi) = 1$, $\theta + \phi + \psi = \frac{1}{2}\pi$ (both carried over from the first part) and $\theta = \phi = \frac{1}{5}\pi$. Since the value of ψ is currently unknown, let us substitute the values of θ and ϕ into (3),

$$\frac{1}{5}\pi + \frac{1}{5}\pi + \psi = \frac{1}{2}\pi \quad (19)$$

$$\Rightarrow \frac{4}{10}\pi + \psi = \frac{5}{10}\pi \tag{20}$$

$$\Rightarrow \psi = \frac{1}{10}\pi \tag{21}$$

Thus, substituting the values of θ, ϕ and ψ into (1) gives us,

$$sin^2\frac{1}{5}\pi + sin^2\frac{1}{5}\pi + sin^2\frac{1}{10}\pi + 2sin(\frac{1}{5}\pi)sin(\frac{1}{5}\pi)sin(\frac{1}{10}\pi) = 1 \tag{22}$$

$$\Rightarrow 2sin^2\frac{1}{5}\pi + sin^2\frac{1}{10}\pi + 2sin^2(\frac{1}{5}\pi)sin(\frac{1}{10}\pi) = 1 \tag{23}$$

Using the double-angle formula $sin(2a) = 2sin(a)cos(a)$ for $2a = \theta = \phi = \frac{1}{5}\pi$ (meaning $a = \frac{1}{10}\pi$),

$$sin(\frac{1}{5}\pi) = sin(\frac{2}{10}\pi) = 2sin(\frac{1}{10}\pi)cos(\frac{1}{10}\pi) \tag{24}$$

$$\Rightarrow sin^2(\frac{1}{5}\pi) = [2sin(\frac{1}{10}\pi)cos(\frac{1}{10}\pi)]^2 = 4sin^2(\frac{1}{10}\pi)cos^2(\frac{1}{10}\pi) \tag{25}$$

$$\Rightarrow 2sin^2(\frac{1}{5}\pi) = 8sin^2(\frac{1}{10}\pi)cos^2(\frac{1}{10}\pi) \tag{26}$$

Substituting (26) into (23) gives us,

$$8sin^2(\frac{1}{10}\pi)cos^2(\frac{1}{10}\pi) + sin^2\frac{1}{10}\pi + sin(\frac{1}{10}\pi)[8sin^2(\frac{1}{10}\pi)cos^2(\frac{1}{10}\pi)] = 1 \tag{27}$$

$$\Rightarrow 8sin^2(\frac{1}{10}\pi)cos^2(\frac{1}{10}\pi) + sin^2\frac{1}{10}\pi + 8sin^3(\frac{1}{10}\pi)cos^2(\frac{1}{10}\pi) = 1 \tag{28}$$

Using $cos^2(\frac{1}{10}\pi) = 1 - sin^2(\frac{1}{10}\pi)$, we can substitute this equation into (28) to get (completely in terms of $sin(\frac{1}{10}\pi)$),

$$8sin^2(\frac{1}{10}\pi)[1 - sin^2(\frac{1}{10}\pi)] + sin^2\frac{1}{10}\pi + 8sin^3(\frac{1}{10}\pi)[1 - sin^2(\frac{1}{10}\pi)] = 1 \tag{29}$$

Expanding the brackets and rearranging the terms in (29) gives us,

$$8sin^2(\frac{1}{10}\pi) - 8sin^4(\frac{1}{10}\pi) + sin^2\frac{1}{10}\pi + 8sin^3(\frac{1}{10}\pi) - 8sin^5(\frac{1}{10}\pi) = 1 \tag{30}$$

$$\Rightarrow 8sin^3(\frac{1}{10}\pi) + 8sin^2(\frac{1}{10}\pi) - 1 - 8sin^5(\frac{1}{10}\pi) - 8sin^4(\frac{1}{10}\pi) + sin^2(\frac{1}{10}\pi) = 0 \tag{31}$$

Factoring out the common expression $8sin^3(\frac{1}{10}\pi) + 8sin^2(\frac{1}{10}\pi) - 1$ in (31) distributes the terms and simplifies the equation,

$$1[8sin^3(\frac{1}{10}\pi)+8sin^2(\frac{1}{10}\pi)-1]-sin^2(\frac{1}{10}\pi)[8sin^3(\frac{1}{10}\pi)+8sin^2(\frac{1}{10}\pi)-1] \tag{32}$$

$$\Rightarrow [1 - sin^2(\frac{1}{10}\pi)][8sin^3(\frac{1}{10}\pi) + 8sin^2(\frac{1}{10}\pi) - 1] = 0 \tag{33}$$

And substituting $cos^2(\frac{1}{10}\pi) = 1 - sin^2(\frac{1}{10}\pi)$ into (33) gives us,

$$cos^2(\frac{1}{10}\pi)[8sin^3(\frac{1}{10}\pi) + 8sin^2(\frac{1}{10}\pi) - 1] = 0 \tag{34}$$

Since $cos^2(\frac{1}{10}\pi)$ can be divided by 0, equation (34) can also be expressed as,

$$8sin^3(\frac{1}{10}\pi) + 8sin^2(\frac{1}{10}\pi) - 1 = 0 \div cos^2(\frac{1}{10}\pi) = 0 \tag{35}$$

Which means that substituting $x = sin(\frac{1}{10}\pi)$ into (35) gives us,

$$8x^3 + 8x^2 - 1 = 0 \tag{36}$$

This proves that $x = sin(\frac{1}{10}\pi)$ satisfies (18) when all the given conditions are true. (Q.E.D.)