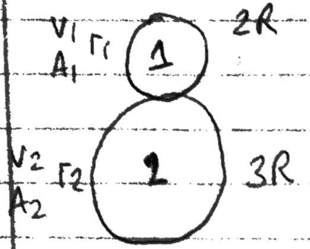
Frosty Puddle - Moosa Saghir



$$\frac{dV_1}{dt} = -kA_1 \Bigg] \text{- given in question}$$

$$V_1 = \frac{4}{3}\pi r_1^3$$

$$\frac{dV_1}{dr_1} = 4\pi r_1^2$$

chain rule

$$\frac{dr_1}{dt} = \frac{dV_1}{dt} \times \frac{dr_1}{dV_1} = \frac{-kA_1}{4\pi r_1^2} = \frac{-k\,4\pi r_1^2}{4\pi r_1^2}$$

$$\frac{dr_1}{dt} = -k$$

solving diff. equation

$$\therefore \quad r_1 = -kt + c$$

$$2R = c \implies r_1 = 2R - kt$$

$$\underbrace{r_2 = 3R - kt}_{}$$

using same logic as we used to find r_1.

total height

$$h = 2r_1 + 2r_2$$

chain rule

$$\frac{dV}{dh} = \cancel{\frac{dh}{dt}} \times \cancel{\frac{dt}{dV}} \quad \frac{dV}{dt} \times \frac{dt}{dh}$$

$$\cancel{\frac{dt}{dV} = \frac{1}{dV/dt}}$$

total volume

$$\frac{dV}{dt} = -k(A_1 + A_2)$$

[~~the overall rate~~ of change of volume $= \frac{dV_1}{dt} + \frac{dV_2}{dt}$]

$$\therefore \frac{dV}{dt} = -k(4\pi r_1^2 + 4\pi r_2^2)$$

$$= -4\pi k(r_1^2 + r_2^2)$$

using expressions for r_1 and r_2 derived earlier

$$= -4\pi k([2R - kt]^2 + [3R - kt]^2)$$

$$= -4\pi k(4R^2 - 4Rkt + k^2t^2 + 9R^2 - 6Rkt + k^2t^2)$$

$$= -4\pi k\,(13R^2 - 10Rkt + 2k^2t^2)$$

$$\cancel{\frac{dt}{dh} = \frac{1}{dh/dt} \qquad \frac{dh}{dt} =}$$

$$h = 2(5R - 2kt)$$
$$= 10R - 4kt$$
$$\Rightarrow \frac{dh}{dt} = -4k \quad \therefore \frac{dt}{dh} = -\frac{1}{4k}$$

$$\therefore \frac{dV}{dh} = \frac{\cancel{-4}\pi \cancel{k}(13R^2 - 10Rkt + 2k^2t^2)}{\cancel{4k}}$$

$$= \pi(13R^2 - 10Rkt + 2k^2t^2)$$

$$h^2 = 100R^2 - 80Rkt + 16k^2t^2$$
$$\therefore \frac{h^2}{8} = 12.5R^2 - 10Rkt + 2k^2t^2$$

~~wuh~~ replacing original expression using $\frac{h^2}{8}$.

$$\therefore \frac{dV}{dh} = \pi\left(\frac{h^2}{8} + \frac{1}{2}R^2\right)$$

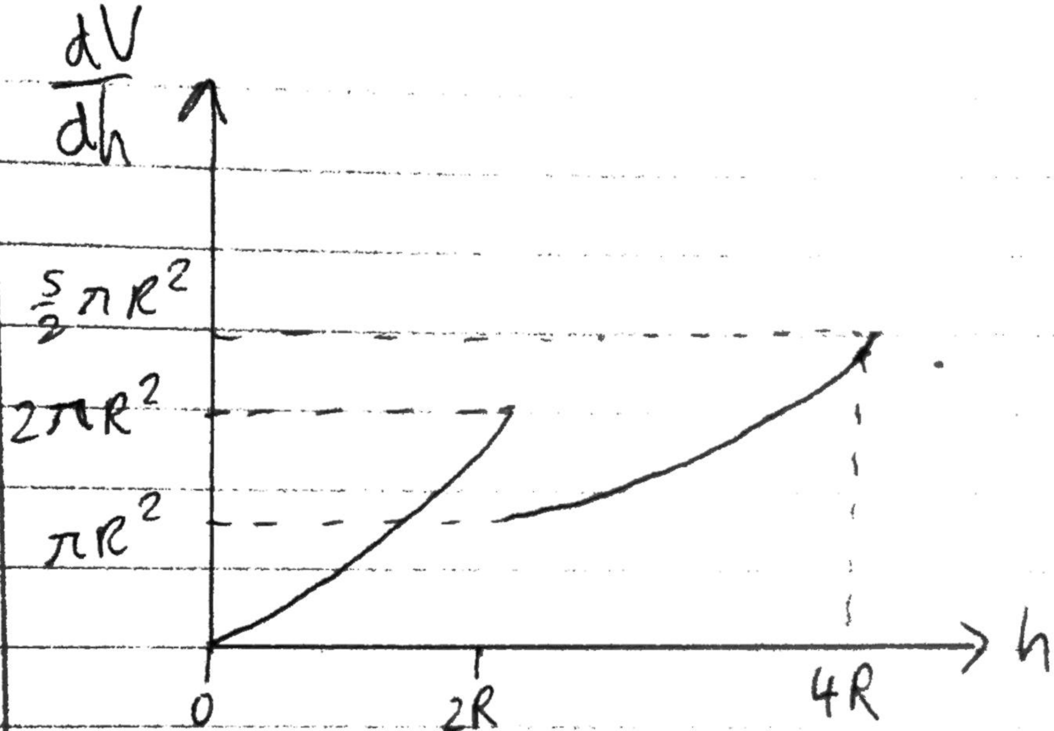
$$\frac{dV}{dh} = \frac{\pi}{8}(h^2 + 4R^2)$$

at $0 \le h < 2R$, top snowball is gone

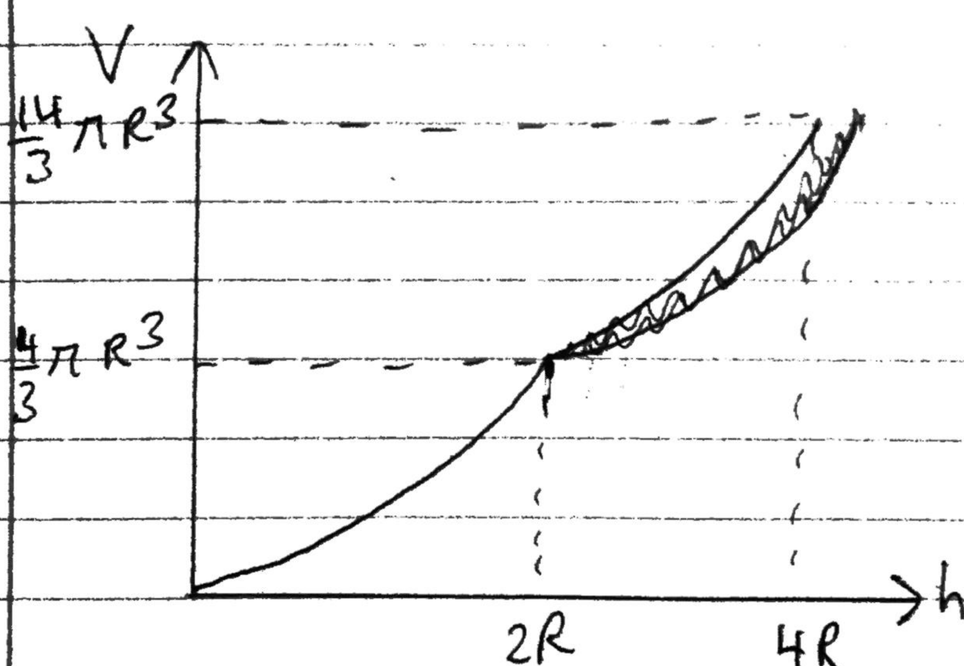
$$\therefore h = 2r_2 \qquad V = \frac{4}{3}\pi r_2^3 \quad \therefore V = \frac{4}{3}\pi\left(\frac{h}{2}\right)^3$$

$$\therefore V = \tfrac{1}{6}\pi h^3 \quad \therefore dV/dh = \tfrac{1}{2}\pi h^2 \qquad 0 \le h < 2R$$

dV



discontinuity is due to change in function at $h = 2R$ when top snowball has melted.



as $r_1 = 2R - kt$

$kt = 2R$ when $r_1 = 0$

$\therefore r_2 = 3R - 2R = R$

$\therefore h = 2R$ when top snowball has melted.