INTO THE WILDERNESS

The argument of a complex number is defined as the angle that the line joining the origin to the point represented by the complex number makes with the real axis and if the complex number is $z=x+yi$, $arg\, z=\arctan\frac{y}{x}$. The modulus of a complex number is the length of the line joining the origin to the point represented by the complex number and if the complex number is $z=x+yi$, $|z|=\sqrt{x^2+y^2}$.

One logical conjecture that comes to mind after experimenting with the Geogebra interactivity is that the angle that the line joining the origin to the point z_3 makes with the real axis ($arg\, z_3$) is the sum of the angles that the lines joining z_1 and z_2 to the origin make with the real axis ($arg\, z_1$ and $arg\, z_2$).

In simpler terms, the conjecture is

$$arg\, z_1+arg\, z_2=arg\, z_3$$

This can be proved using a variety of methods

Let $z_1=a+bi$ and $z_2=c+di$. Since $z_3=z_1z_2$, $z_3=(a+bi)(c+di)$, therefore

$z_3=ac-bd+(ad+bc)i$. $arg\, z_1=\arctan\frac{b}{a}$ and $arg\, z_2=\arctan\frac{d}{c}$ and

$arg\, z_3=\arctan\frac{ad+bc}{ac-bd}$. So now we must prove that

$$\arctan\frac{b}{a}+\arctan\frac{d}{c}=\arctan\frac{ad+bc}{ac-bd}$$

Let $\arctan\frac{b}{a}=\theta \rightarrow \tan\theta=\frac{b}{a}$ and let $\arctan\frac{d}{c}=\phi \rightarrow \tan\phi=\frac{d}{c}$. So we have

$$\theta+\phi=\arctan\frac{ad+bc}{ac-bd}$$

Taking the tangent of both sides, we now have to prove the following

$$\tan(\theta+\phi)=\frac{ad+bc}{ac-bd}$$

LHS:

$$\tan(\theta+\phi)$$

Using trig identities, this is equal to

$$\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

Substituting in the values of $\tan\theta$ and $\tan\phi$ gives

$$\frac{\frac{b}{a}+\frac{d}{c}}{1-\frac{bd}{ac}}$$

Multiplying the numerator and denominator by ac , we get

$$\frac{bc+ad}{ac-bd}=RHS$$

Therefore, we have proved that, if $z_1z_2=z_3$, $arg\, z_1+arg\, z_2=arg\, z_3$

Another logical conjecture that came to mind after some time with the interactive and doing various calculations was, if z_3 is the product of z_1 and z_2, then the modulus of z_3 is the product of the modulus of z_1 and the modulus of z_2.

The following is a proof of the statement $|z_3|=|z_1||z_2|$

LHS:

$|z_3|=|(ac-bd)+(ad+bc)i|$

By the definition of the modulus of a complex number,

$|z_3|=\sqrt{(ac-bd)^2+(ad+bc)^2}$

Expanding the brackets,

$|z_3|=\sqrt{a^2c^2-2abcd+b^2d^2+a^2d^2+2abcd+b^2c^2}$

Gathering like terms,

$|z_3|=\sqrt{a^2c^2+b^2c^2+a^2d^2+b^2d^2}$

Factorising,

$|z_3|=\sqrt{c^2(a^2+b^2)+d^2(a^2+b^2)}$

Factorising again,

$|z_3|=\sqrt{(a^2+b^2)(c^2+d^2)}$

Using $(ab)^n=a^n b^n$,

$|z_3|=\sqrt{a^2+b^2}\sqrt{c^2+d^2}$

And since $|z_1|=\sqrt{a^2+b^2}$ and $|z_2|=\sqrt{c^2+d^2}$,

$|z_3|=|z_1||z_2|$

The following are much shorter methods to prove the conjectures.

If $\theta=arg\,z$ and $|z|=r$ (the modulus of z), z can be represented by $z=r(\cos\theta+\sin\theta)$. Using Maclaurin expansions of e^x, $\cos x$ and $\sin x$, it can be proved that $e^{ix}=\cos x+i\sin x$, therefore any complex number can be written as $z=re^{i\theta}$

Now, if we let $z_1=r_1e^{i\theta_1}$ and $z_2=r_2e^{i\theta_2}$, since $z_3=z_1z_2$, $z_3=(r_1e^{i\theta_1})(r_2e^{i\theta_2})$ and, using the laws of indices, we have $z_3=r_1r_2e^{i(\theta_1+\theta_2)}$

So now, we see that $arg\,z_3=\theta_1+\theta_2=arg\,z_1+arg\,z_2$ and $|z_3|=|z_1||z_2|$

After investigation using the interactivity, the line L joining the point represented by z_1 and the point represented by z_3 passes through the origin when the imaginary part of z_2 is 0. Therefore, $z_2=c$, where c is real. Since $z_3=z_1z_2$, $z_3=(a+bi)(c)$ $\rightarrow z_3=ac+bci$. Let L_1 be the line joining z_1 to the origin and L_3 is defined similarly. L passes through the origin if and only if the gradient of

L_1 is equal to the gradient of L_3. The gradient of L_1 is $\frac{b-0}{a-0}=\frac{b}{a}$ and the gradient of L_3 is $\frac{bc-0}{ac-0}=\frac{bc}{ac}=\frac{b}{a}$
therefore the gradient of L_1 is equal to the gradient of L_3, and so L passes through the origin when the imaginary part of z_2 is 0.

z_2 and z_3 are equidistant from the origin if and only if $r_2=r_1 r_2$ $\rightarrow r_1=1$, i.e if and only if $z_1=a+bi$, where $a^2+b^2=1$ or $z_1=e^{i\theta}$.