If $z_1 = a+bi$, $z_2 = c+di$ and $z_3 = z_1 z_2$, then

$$z_3 = (a+bi)(c+di) = ac - bd + (ad+bc)i$$

z_3 is real if and only if $ad+bc=0$.

So if, given $z_1 = a+bi$, one wants to choose z_2 such that the product of z_1 and z_2 is a real number, choose $z_2 = c+di$ such that $ad+bc=0$.

If, given a $z_1 = a+bi$, one wants to choose z_2 such that the product of z_1 and z_2 is purely imaginary, choose $z_2 = c+di$ such that $ac-bd=0$.

The following is a method for finding the complex number $c+di$, such that $(a+bi)(c+di) = x+yi$.

Expanding the LHS,
$(a+bi)(c+di) = ac + adi + bci - bd$
Gathering like terms, this is equal to
$ac - bd + (ad+bc)i$

Now since $\alpha_1 + \beta_1 i = \alpha_2 + \beta_2 i$ if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$,
$ac - bd + (ad+bc)i = x+yi$ if and only if
$ac - bd = x$
and
$ad + bc = y$

These equations can be solved simultaneously for c and d , revealing that
$c = \frac{ax+by}{a^2+b^2}$ and $d = \frac{ay-bx}{a^2+b^2}$.
So one must multiply $a+bi$ by $\frac{ax+by}{a^2+b^2} + \frac{ay-bx}{a^2+b^2} i$ to obtain $x+yi$.