STEP Support Programme

Statistics Questions: Solutions

2010 S1 Q12

1 The one piece of information you need here (other than GSCE probability) is the definition of $\mathrm{E}(X)$:

$$\mathrm{E}(X) = \sum n \times \mathrm{P}(X = n)$$

The expectation is given by

$$\begin{array}{rcccccccccc} \mathrm{E}(X) & = & \mathrm{P}(X=1) & + & 2\times\mathrm{P}(X=2) & + & 3\times\mathrm{P}(X=3) & + & 4\times\mathrm{P}(X=4) & + & \cdots \\ & = & \mathrm{P}(X=1) & + & \mathrm{P}(X=2) & + & \mathrm{P}(X=3) & + & \mathrm{P}(X=4) & + & \cdots \\ & & & + & \mathrm{P}(X=2) & + & \mathrm{P}(X=3) & + & \mathrm{P}(X=4) & + & \cdots \\ & & & & & + & \mathrm{P}(X=3) & + & \mathrm{P}(X=4) & + & \cdots \\ & & & & & & & + & \mathrm{P}(X=4) & + & \cdots \end{array}$$

Then the top row is $\mathrm{P}(X \geqslant 1)$, then second row is $\mathrm{P}(X \geqslant 2)$ etc.

If $X \geqslant 4$ then the first 3 boxes were either all daddy penguins or all mummy penguins. Therefore we have $\mathrm{P}(X \geqslant 4) = p \times p \times p + q \times q \times q = p^3 + q^3$.

Similarly $\mathrm{P}(X \geqslant n) = p^{n-1} + q^{n-1}$, as in this case the first $n-1$ boxes are all daddy penguins or are all mummy penguins, but only for $n \geqslant 2$.

We have $\mathrm{P}(X \geqslant 1) = 1$ which is not the same as $p^0 + q^0 = 2$.

Using the expression for expectation given at the beginning of the question you have:

$$\mathrm{E}(X) = 1 + \sum_{n=2}^{\infty} \left(p^{n-1} + q^{n-1}\right) .$$

We can split the sum up and use the sum to infinity of a geometric sequence to get:

$$\mathrm{E}(X) = 1 + \frac{p}{1-p} + \frac{q}{1-q} .$$

Noting that $p + q = 1$ means we can write this as:

$$\mathrm{E}(X) = 1 + \frac{p}{q} + \frac{q}{p} = 1 + \frac{p^2+q^2}{pq} .$$

This is not quite in the correct form, so we try fiddling with it:

$$\begin{aligned} \mathrm{E}(X) &= \frac{p^2+q^2+pq}{pq} \\ &= \frac{p^2+q^2+2pq-pq}{pq} \\ &= \frac{(p+q)^2}{pq} - \frac{pq}{pq} \\ &= \frac{1}{pq} - 1 \qquad \textit{as required.} \end{aligned}$$

Alternatively, we can replace 1 with $p^0 + q^0 - 1$. This gives:

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{n=1}^{\infty} p^{n-1} + \sum_{n=1}^{\infty} q^{n-1} - 1 \\
&= \frac{1}{1-p} + \frac{1}{1-q} - 1 \\
&= \frac{1}{q} + \frac{1}{p} - 1 \\
&= \frac{p+q}{pq} - 1 \\
&= \frac{1}{pq} - 1\,.
\end{aligned}$$

For the last part, write the expectation as $\mathrm{E}(X) = \dfrac{1}{p(1-p)} - 1$.

You can find the minimum value of $y = \dfrac{1}{x(1-x)} - 1$, but it is easier to note that the minimum value of $\mathrm{E}(X)$ corresponds to the maximum value of $p(1-p)$. The maximum stationary point of $y = x(1-x)$ is at $\left(\frac{1}{2}, \frac{1}{4}\right)$, and so we have $p(1-p) \leqslant \frac{1}{4}$. Hence we have $\mathrm{E}(X) \geqslant 4 - 1 = 3$.

2013 S2 Q12

2 Here we need the definitions of $\mathrm{E}(X)$, $\mathrm{Var}\,(X)$, the probabilities of the Poisson distribution $\mathrm{P}(U=r)=\dfrac{\mathrm{e}^{-\lambda}\lambda^r}{r!}$ and the variance of the Poisson distribution, $\mathrm{Var}\,(U)=\lambda$. Everything else is manipulating sums and equations.

(i) We have:

$$\begin{aligned}\mathrm{E}(X) &= 1\times\frac{\mathrm{e}^{-\lambda}\lambda^1}{1!}+3\times\frac{\mathrm{e}^{-\lambda}\lambda^3}{3!}+5\times\frac{\mathrm{e}^{-\lambda}\lambda^5}{5!}+\dots\\ &= \mathrm{e}^{-\lambda}\lambda^1+\cancel{3}\times\frac{\mathrm{e}^{-\lambda}\lambda^3}{\cancel{3}\times 2!}+\cancel{5}\times\frac{\mathrm{e}^{-\lambda}\lambda^5}{\cancel{5}\times 4!}+\dots\\ &= \mathrm{e}^{-\lambda}\lambda\left(1+\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}+\dots\right)\\ &= \mathrm{e}^{-\lambda}\lambda\alpha\,.\end{aligned}$$

Similarly:

$$\begin{aligned}\mathrm{E}(Y) &= 2\times\frac{\mathrm{e}^{-\lambda}\lambda^2}{2!}+4\times\frac{\mathrm{e}^{-\lambda}\lambda^4}{4!}+6\times\frac{\mathrm{e}^{-\lambda}\lambda^6}{6!}+\dots\\ &= \mathrm{e}^{-\lambda}\lambda\left(\lambda+\frac{\lambda^3}{3!}+\frac{\lambda^5}{5!}+\dots\right)\\ &= \mathrm{e}^{-\lambda}\lambda\beta\,.\end{aligned}$$

(ii) We have $\mathrm{Var}\,(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$. First find $\mathrm{E}(X^2)$:

$$\begin{aligned}\mathrm{E}(X^2) &= 1^2\times\frac{\mathrm{e}^{-\lambda}\lambda^1}{1!}+3^2\times\frac{\mathrm{e}^{-\lambda}\lambda^3}{3!}+5^2\times\frac{\mathrm{e}^{-\lambda}\lambda^5}{5!}+\dots\\ &= \mathrm{e}^{-\lambda}\lambda^1+\cancel{3}\times 3\times\frac{\mathrm{e}^{-\lambda}\lambda^3}{\cancel{3}\times 2!}+\cancel{5}\times 5\times\frac{\mathrm{e}^{-\lambda}\lambda^5}{\cancel{5}\times 4!}+\dots\\ &= \mathrm{e}^{-\lambda}\lambda\left(1+\frac{3\lambda^2}{2!}+\frac{5\lambda^4}{4!}+\dots\right)\\ &= \mathrm{e}^{-\lambda}\lambda\left(1+\frac{(1+2)\lambda^2}{2!}+\frac{(1+4)\lambda^4}{4!}+\dots\right)\\ &= \mathrm{e}^{-\lambda}\lambda\left(1+\frac{\lambda^2}{2!}+\frac{\cancel{2}\lambda^2}{\cancel{2}\times 1!}+\frac{\lambda^4}{4!}+\frac{\cancel{4}\lambda^4}{\cancel{4}\times 3!}+\dots\right)\\ &= \mathrm{e}^{-\lambda}\lambda\left(1+\frac{\lambda^2}{2!}+\frac{\lambda^4}{4!}+\dots+\lambda\left[\frac{\lambda}{1!}+\frac{\lambda^3}{3!}+\dots\right]\right)\\ &= \mathrm{e}^{-\lambda}\lambda\,(\alpha+\lambda\beta)\end{aligned}$$

Then we have:

$$\begin{aligned}\text{Var}\,(X) &= \text{E}(X^2) - [\text{E}(X)]^2 \\ &= \text{e}^{-\lambda}\lambda\,(\alpha + \lambda\beta) - \left(\text{e}^{-\lambda}\lambda\alpha\right)^2\end{aligned}$$

which is not quite the required result. However, we have:

$$\alpha + \beta = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \ldots = \text{e}^{\lambda}$$

and hence $\text{e}^{-\lambda} = \dfrac{1}{\alpha+\beta}$. This gives $\text{Var}\,(X) = \dfrac{\lambda\alpha + \lambda^2\beta}{\alpha+\beta} - \dfrac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$.

The same approach gives $\text{Var}\,(Y) = \dfrac{\lambda\beta + \lambda^2\alpha}{\alpha+\beta} - \dfrac{\lambda^2\beta^2}{(\alpha+\beta)^2}$.

For the last part, start by noting that $\text{Var}\,(X+Y) = \text{Var}\,(U) = \lambda$. We then want to find non-zero values of λ for which:

$$\frac{\lambda\alpha + \lambda^2\beta}{\alpha+\beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2} + \frac{\lambda\beta + \lambda^2\alpha}{\alpha+\beta} - \frac{\lambda^2\beta^2}{(\alpha+\beta)^2} = \lambda\,.$$

Then either $\lambda = 0$, or:

$$\begin{aligned}(\alpha + \lambda\beta)(\alpha+\beta) - \lambda\alpha^2 + (\beta + \lambda\alpha)(\alpha+\beta) - \lambda\beta^2 &= (\alpha+\beta)^2 \\ \cancel{\alpha^2} + \cancel{\alpha\beta} + \lambda\alpha\beta + \cancel{\lambda\beta^2} - \cancel{\lambda\alpha^2} + \cancel{\alpha\beta} + \cancel{\beta^2} + \cancel{\lambda\alpha^2} + \lambda\alpha\beta - \cancel{\lambda\beta^2} &= \cancel{\alpha^2} + \cancel{2\alpha\beta} + \cancel{\beta^2} \\ 2\lambda\alpha\beta &= 0\end{aligned}$$

If $\lambda \neq 0$ this can only be solved if one of α and β is zero. Since $\alpha > 0$ and $\beta > 0$ there are no non-zero values of λ for which $\text{Var}\,(X) + \text{Var}\,(Y) = \text{Var}\,(X+Y)$.