

STEP Support Programme

Statistics Questions: Solutions

2010 S1 Q12

- 1 The one piece of information you need here (other than GSCE probability) is the definition of $E(X)$:

$$E(X) = \sum n \times P(X = n)$$

The expectation is given by

$$\begin{aligned} E(X) &= P(X = 1) + 2 \times P(X = 2) + 3 \times P(X = 3) + 4 \times P(X = 4) + \dots \\ &= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + \dots \\ &\quad + P(X = 2) + P(X = 3) + P(X = 4) + \dots \\ &\quad + P(X = 3) + P(X = 4) + \dots \\ &\quad + P(X = 4) + \dots \end{aligned}$$

Then the top row is $P(X \geq 1)$, then second row is $P(X \geq 2)$ etc.

If $X \geq 4$ then the first 3 boxes were either all daddy penguins or all mummy penguins. Therefore we have $P(X \geq 4) = p \times p \times p + q \times q \times q = p^3 + q^3$.

Similarly $P(X \geq n) = p^{n-1} + q^{n-1}$, as in this case the first $n-1$ boxes are all daddy penguins or are all mummy penguins, but only for $n \geq 2$.

We have $P(X \geq 1) = 1$ which is not the same as $p^0 + q^0 = 2$.

Using the expression for expectation given at the beginning of the question you have:

$$E(X) = 1 + \sum_{n=2}^{\infty} (p^{n-1} + q^{n-1}) .$$

We can split the sum up and use the sum to infinity of a geometric sequence to get:

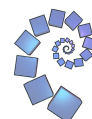
$$E(X) = 1 + \frac{p}{1-p} + \frac{q}{1-q} .$$

Noting that $p + q = 1$ means we can write this as:

$$E(X) = 1 + \frac{p}{q} + \frac{q}{p} = 1 + \frac{p^2 + q^2}{pq} .$$

This is not quite in the correct form, so we try fiddling with it:

$$\begin{aligned} E(X) &= \frac{p^2 + q^2 + pq}{pq} \\ &= \frac{p^2 + q^2 + 2pq - pq}{pq} \\ &= \frac{(p+q)^2}{pq} - \frac{pq}{pq} \\ &= \frac{1}{pq} - 1 \quad \text{as required.} \end{aligned}$$

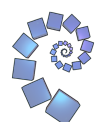


Alternatively, we can replace 1 with $p^0 + q^0 - 1$. This gives:

$$\begin{aligned}
 E(X) &= \sum_{n=1}^{\infty} p^{n-1} + \sum_{n=1}^{\infty} q^{n-1} - 1 \\
 &= \frac{1}{1-p} + \frac{1}{1-q} - 1 \\
 &= \frac{1}{q} + \frac{1}{p} - 1 \\
 &= \frac{p+q}{pq} - 1 \\
 &= \frac{1}{pq} - 1.
 \end{aligned}$$

For the last part, write the expectation as $E(X) = \frac{1}{p(1-p)} - 1$.

You can find the minimum value of $y = \frac{1}{x(1-x)} - 1$, but it is easier to note that the minimum value of $E(X)$ corresponds to the maximum value of $p(1-p)$. The maximum stationary point of $y = x(1-x)$ is at $(\frac{1}{2}, \frac{1}{4})$, and so we have $p(1-p) \leq \frac{1}{4}$. Hence we have $E(X) \geq 4 - 1 = 3$.



2013 S2 Q12

- 2** Here we need the definitions of $E(X)$, $\text{Var}(X)$, the probabilities of the Poisson distribution $P(U = r) = \frac{e^{-\lambda}\lambda^r}{r!}$ and the variance of the Poisson distribution, $\text{Var}(U) = \lambda$. Everything else is manipulating sums and equations.

(i) We have:

$$\begin{aligned} E(X) &= 1 \times \frac{e^{-\lambda}\lambda^1}{1!} + 3 \times \frac{e^{-\lambda}\lambda^3}{3!} + 5 \times \frac{e^{-\lambda}\lambda^5}{5!} + \dots \\ &= e^{-\lambda}\lambda^1 + \cancel{3} \times \frac{e^{-\lambda}\lambda^3}{\cancel{3} \times 2!} + \cancel{5} \times \frac{e^{-\lambda}\lambda^5}{\cancel{5} \times 4!} + \dots \\ &= e^{-\lambda}\lambda \left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots \right) \\ &= e^{-\lambda}\lambda\alpha. \end{aligned}$$

Similarly:

$$\begin{aligned} E(Y) &= 2 \times \frac{e^{-\lambda}\lambda^2}{2!} + 4 \times \frac{e^{-\lambda}\lambda^4}{4!} + 6 \times \frac{e^{-\lambda}\lambda^6}{6!} + \dots \\ &= e^{-\lambda}\lambda \left(\lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \dots \right) \\ &= e^{-\lambda}\lambda\beta. \end{aligned}$$

(ii) We have $\text{Var}(X) = E(X^2) - [E(X)]^2$. First find $E(X^2)$:

$$\begin{aligned} E(X^2) &= 1^2 \times \frac{e^{-\lambda}\lambda^1}{1!} + 3^2 \times \frac{e^{-\lambda}\lambda^3}{3!} + 5^2 \times \frac{e^{-\lambda}\lambda^5}{5!} + \dots \\ &= e^{-\lambda}\lambda^1 + \cancel{3} \times 3 \times \frac{e^{-\lambda}\lambda^3}{\cancel{3} \times 2!} + \cancel{5} \times 5 \times \frac{e^{-\lambda}\lambda^5}{\cancel{5} \times 4!} + \dots \\ &= e^{-\lambda}\lambda \left(1 + \frac{3\lambda^2}{2!} + \frac{5\lambda^4}{4!} + \dots \right) \\ &= e^{-\lambda}\lambda \left(1 + \frac{(1+2)\lambda^2}{2!} + \frac{(1+4)\lambda^4}{4!} + \dots \right) \\ &= e^{-\lambda}\lambda \left(1 + \frac{\lambda^2}{2!} + \frac{\cancel{2}\lambda^2}{\cancel{2} \times 1!} + \frac{\lambda^4}{4!} + \frac{\cancel{4}\lambda^4}{\cancel{4} \times 3!} + \dots \right) \\ &= e^{-\lambda}\lambda \left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \dots + \lambda \left[\frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \dots \right] \right) \\ &= e^{-\lambda}\lambda(\alpha + \lambda\beta) \end{aligned}$$



Then we have:

$$\begin{aligned}\text{Var}(X) &= E(X^2) - [E(X)]^2 \\ &= e^{-\lambda} \lambda (\alpha + \lambda \beta) - (e^{-\lambda} \lambda \alpha)^2\end{aligned}$$

which is not quite the required result. However, we have:

$$\alpha + \beta = 1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \dots = e^\lambda$$

and hence $e^{-\lambda} = \frac{1}{\alpha + \beta}$. This gives $\text{Var}(X) = \frac{\lambda\alpha + \lambda^2\beta}{\alpha + \beta} - \frac{\lambda^2\alpha^2}{(\alpha + \beta)^2}$.

The same approach gives $\text{Var}(Y) = \frac{\lambda\beta + \lambda^2\alpha}{\alpha + \beta} - \frac{\lambda^2\beta^2}{(\alpha + \beta)^2}$.

For the last part, start by noting that $\text{Var}(X + Y) = \text{Var}(U) = \lambda$. We then want to find non-zero values of λ for which:

$$\frac{\lambda\alpha + \lambda^2\beta}{\alpha + \beta} - \frac{\lambda^2\alpha^2}{(\alpha + \beta)^2} + \frac{\lambda\beta + \lambda^2\alpha}{\alpha + \beta} - \frac{\lambda^2\beta^2}{(\alpha + \beta)^2} = \lambda.$$

Then either $\lambda = 0$, or:

$$\begin{aligned}(\alpha + \lambda\beta)(\alpha + \beta) - \lambda\alpha^2 + (\beta + \lambda\alpha)(\alpha + \beta) - \lambda\beta^2 &= (\alpha + \beta)^2 \\ \cancel{\alpha^2} + \cancel{\alpha\beta} + \lambda\alpha\beta + \cancel{\lambda\beta^2} - \cancel{\lambda\alpha^2} + \cancel{\alpha\beta} + \beta^2 + \cancel{\lambda\alpha^2} + \lambda\alpha\beta - \cancel{\lambda\beta^2} &= \cancel{\alpha^2} + 2\alpha\beta + \cancel{\beta^2} \\ 2\lambda\alpha\beta &= 0\end{aligned}$$

If $\lambda \neq 0$ this can only be solved if one of α and β is zero. Since $\alpha > 0$ and $\beta > 0$ there are no non-zero values of λ for which $\text{Var}(X) + \text{Var}(Y) = \text{Var}(X + Y)$.

