STEP Support Programme

STEP II Pure Questions: Solutions

1 **(i)** With the given substitution we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \left(1+x^2\right)^{\frac{1}{2}} \times \dfrac{\mathrm{d}u}{\mathrm{d}x} + xu\left(1+x^2\right)^{-\frac{1}{2}}$.

This should reduce the differential equation to one where the variables are separable. The integral in x can be tackled with a substitution, or by inspection ("guessing" what the answer is and then checking it). You should find that $-\dfrac{1}{u} = \frac{1}{3}\left(1+x^2\right)^{\frac{1}{2}} + c$.

You do need to give the final answer in terms of y, and use the initial condition. The final answer is:

$$y = \frac{3\left(1+x^2\right)^{\frac{1}{2}}}{4-(1+x^2)^{\frac{3}{2}}}\,.$$

(ii) This time use $y = u(1+x^3)^{\frac{1}{3}}$. The final answer is:

$$y = \frac{4\left(1+x^3\right)^{\frac{1}{3}}}{5-(1+x^3)^{\frac{4}{3}}}.$$

(iii) This part requires you to look at the cases $n=2$ (part **(i)**) and $n=3$ (part **(iii)**) and generalise in terms of n. Final answer:

$$y = \frac{(n+1)\left(1+x^n\right)^{\frac{1}{n}}}{(n+2)-\left(1+x^n\right)^{\frac{n+1}{n}}}.$$

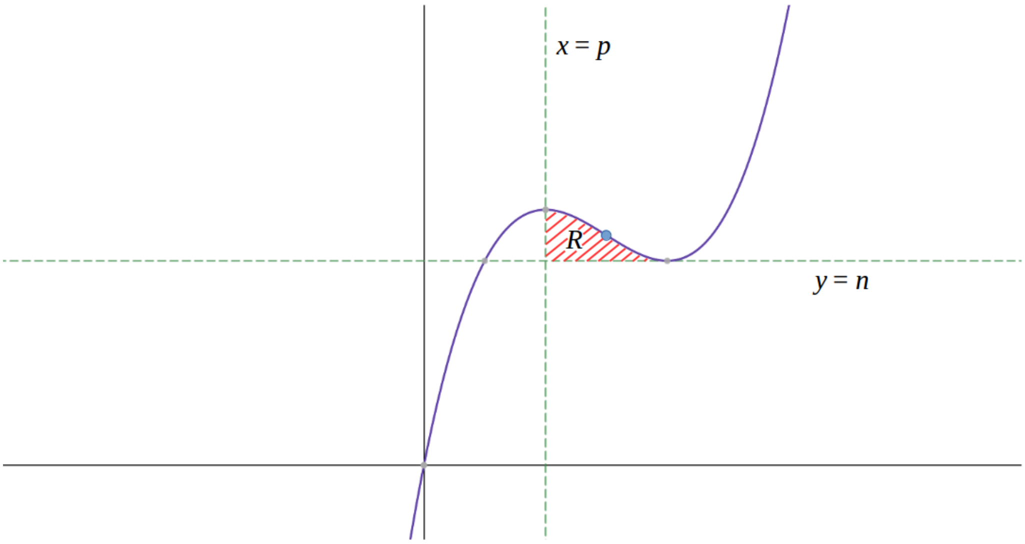
Note: in each case equivalent forms would be acceptable, but it is easier to find the general case if you simplify the $n=2$ and $n=3$ cases as shown.

2 This question caused some problems as the region "R" is not uniquely defined - there are two possible regions! When it was set, all those who picked the "wrong" region had their work marked by the Principal Examiner to make sure that they were not disadvantaged.

(i) Differentiation gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 - 2bx + c \equiv k(x-p)(x-q)$. By equating coefficients we get $k = 6$, $b = 3(p+q)$ and $c = 6pq$.

(ii) This is a cubic, which passes through $(0,0)$. The conditions $p > 0$ and $n > 0$ mean that both of the turning points lie in the first quadrant. The point of inflection lies half way between the turning points[1]. Note that this is a *non-stationary* point of inflection, i.e. the gradient of the curve is not zero here.

The sketch should look something like:



The curve has rotational symmetry order 2 about the point of inflection.

(iii) $m - n = (2p^3 - bp^2 + cp) - (2q^3 - bq^2 + cq) = 2(p^3 - q^3) - b(p^2 - q^2) + c(p-q)$. The best starting point is probably to factorise out $(p-q)$ — or even $(q-p)$ — and then substitute in your answers for b and c.

The difference of two cubes identity, $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$, will be helpful (and is well worth knowing).

(iv) The last bit uses the fact that the shaded is region is half of a rectangle which is $p - q$ wide and $m - n$ high, along with the result from part **(iii)**.

[1] It is not necessary to do so, but if you set the second derivative equal to zero you will find that the x coordinate of the point of point of inflection is $x = \frac{b}{6} = \frac{p+q}{2}$, i.e. half way between $x = p$ and $x = q$.

3 **(i)** Differentiation gives $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 3q$ so the stationary points satisfy $x^2 = q$ and are at $(\sqrt{q},\, -2q\sqrt{q} - q(1+q))$ and $(-\sqrt{q},\, 2q\sqrt{q} - q(1+q))$. The y coordinate of the first of these is obviously negative if $q > 0$ (but this still should be stated!). The y coordinate of the second one of these can be written as $-q(1 + q - 2\sqrt{q}) = -q\left(1 - \sqrt{q}\right)^2$ and so this is also negative (since we are told that $q \neq 1$, otherwise this point would be on the x axis and there would be two points of intersection of the curve with the x axis). Hence both turning points of the cubic lie below the x axis and the curve only crosses the x axis once.

(ii) The equation in u is $(u^3)^2 - q(1+q)u^3 + q^3 = 0$. Solving for u^3 gives:

$$u^3 = \frac{q(1+q) \pm \sqrt{q^2(1+q)^2 - 4q^3}}{2}\,.$$

The part in the square brackets is equal to $q^4 - 2q^3 + q^2 = q^2(1-q)^2$, and so we have $u^3 = q$ or q^2 i.e. $u = q^{\frac{1}{3}}$ or $q^{\frac{2}{3}}$. Both of these values of u give the same value of x, which is good as we have shown that there is only one possible value in part **(i)**, i.e. $x = q^{\frac{1}{3}} + q^{\frac{2}{3}}$.

(iii) Since $t^2 - pt + q \equiv (t-\alpha)(t-\beta)$ we have $\alpha\beta = q$ and $\alpha + \beta = p$.

We also have $(\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3\alpha^2\beta + 3\alpha\beta^2 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$.
This means that $p^3 = \alpha^3 + \beta^3 + 3qp$.

Since one of the roots is the square of the other we know that either $\alpha^2 = \beta$ or $\beta^2 = \alpha$. Hence we have $(\alpha^2 - \beta)(\beta^2 - \alpha) = 0$. Hence $\alpha^2\beta^2 + \alpha\beta - \alpha^3 - \beta^3 = 0$ and so $q^2 + q - (p^3 - 3qp) = 0$. This can be written as $p^3 - 3qp - q(1+q) = 0$ which looks suspiciously like something that appears in parts **(i)** and **(ii)**. Then with the given conditions on q we have the same situation as in part **(ii)** and so $p = q^{\frac{1}{3}} + q^{\frac{2}{3}}$.

4 Note that you cannot just write down the result from the formula book here!

Using the given substitution we have:

$$\begin{aligned}\int \frac{1}{a^2 + a^2\tan^2\theta} \times a\sec^2\theta \, \mathrm{d}\theta &= \int \frac{1}{a}\mathrm{d}\theta \\ &= \frac{1}{a} \times \theta + c \\ &= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c\,.\end{aligned}$$

Since this result is in the "stem" of the question, you should expect to use it at least once (and probably more often) in the following question parts.

(i) **(a)** Using the substitution $t = \sin x$ gives us:

$$\begin{aligned}\int_0^{\frac{1}{2}\pi} \frac{\cos x}{1 + \sin^2 x}\,\mathrm{d}x &= \int_0^1 \frac{\cos x}{1+t^2}\,\frac{\mathrm{d}t}{\cos x} \\ &= \int_0^1 \frac{1}{1+t^2}\,\mathrm{d}t \\ &= \big[\arctan t\big]_0^1 \qquad \text{using the stem result} \\ &= \frac{\pi}{4}\end{aligned}$$

(b) Using the suggested substitution (which means that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\sec^2\frac{1}{2}x$) gives us:

$$\begin{aligned}\int_0^1 \frac{1-t^2}{1+6t^2+t^4}\,\mathrm{d}t &= \int_0^{\frac{\pi}{2}} \frac{1-\tan^2\frac{1}{2}x}{1+6\tan^2\frac{1}{2}x+\tan^4\frac{1}{2}x} \times \tfrac{1}{2}\sec^2\tfrac{1}{2}x\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\tan^2\frac{1}{2}x}{\left(1+6\tan^2\frac{1}{2}x+\tan^4\frac{1}{2}x\right)\cos^2\frac{1}{2}x}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1-\dfrac{\sin^2\frac{1}{2}x}{\cos^2\frac{1}{2}x}}{\left(\cos^2\frac{1}{2}x+6\sin^2\frac{1}{2}x+\dfrac{\sin^4\frac{1}{2}x}{\cos^2\frac{1}{2}x}\right)}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}{\left(\cos^4\frac{1}{2}x+6\sin^2\frac{1}{2}x\cos^2\frac{1}{2}x+\sin^4\frac{1}{2}x\right)}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}{\left(\cos^2\frac{1}{2}x+\sin^2\frac{1}{2}x\right)^2+4\sin^2\frac{1}{2}x\cos^2\frac{1}{2}x}\,\mathrm{d}x \\ &= \tfrac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,\mathrm{d}x = \tfrac{1}{2}I\end{aligned}$$

The last step came from using $\cos 2A = \cos^2 A - \sin^2 A$, $\sin 2A = 2\sin A\cos A$ and $\cos^2 A + \sin^2 A = 1$.

(ii) Using the substitution $t = \tan \frac{1}{2}x$ in the same way as in part **(i)(b)** results in the integral $\dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} \dfrac{\cos x}{1 + 3\sin^2 x}\,\mathrm{d}x$.

A further substitution of $y = \sqrt{3}\sin x$ gives the integral:

$$\frac{1}{2\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{1+y^2}\,\mathrm{d}y = \frac{1}{2\sqrt{3}}\Big[\arctan y\Big]_0^{\sqrt{3}}$$

which gives the final answer as $\dfrac{1}{2\sqrt{3}} \times \dfrac{\pi}{3} = \dfrac{1}{6\sqrt{3}}\pi$.