STEP Support Programme

STEP II Pure Questions

1 **(i)** By writing $y = u(1+x^2)^{\frac{1}{2}}$, where u is a function of x, find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = xy + \frac{x}{1+x^2}$$

for which $y = 1$ when $x = 0$.

(ii) Find the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y + \frac{x^2}{1+x^3}$$

for which $y = 1$ when $x = 0$.

(iii) Give, without proof, a conjecture for the solution of the equation

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^{n-1}y + \frac{x^{n-1}}{1+x^n}$$

for which $y = 1$ when $x = 0$, where n is an integer greater than 1.

2 A curve has equation $y = 2x^3 - bx^2 + cx$. It has a maximum point at (p, m) and a minimum point at (q, n) where $p > 0$ and $n > 0$. Let R be the region enclosed by the curve, the line $x = p$ and the line $y = n$.

(i) Express b and c in terms of p and q.

(ii) Sketch the curve. Mark on your sketch the point of inflection and shade the region R. Describe the symmetry of the curve.

(iii) Show that $m - n = (q-p)^3$.

(iv) Show that the area of R is $\frac{1}{2}(q-p)^4$.

3 **(i)** By considering the positions of its turning points, show that the curve with equation

$$y = x^3 - 3qx - q(1+q)\,,$$

where $q > 0$ and $q \neq 1$, crosses the x-axis once only.

(ii) Given that x satisfies the cubic equation

$$x^3 - 3qx - q(1+q) = 0\,,$$

and that

$$x = u + q/u\,,$$

obtain a quadratic equation satisfied by u^3. Hence find the real root of the cubic equation in the case $q > 0$, $q \neq 1$.

(iii) The quadratic equation

$$t^2 - pt + q = 0$$

has roots α and β. Show that

$$\alpha^3 + \beta^3 = p^3 - 3qp\,.$$

It is given that one of these roots is the square of the other. By considering the expression $(\alpha^2 - \beta)(\beta^2 - \alpha)$, find a relationship between p and q. Given further that $q > 0$, $q \neq 1$ and p is real, determine the value of p in terms of q.

4 By writing $x = a\tan\theta$, show that, for $a \neq 0$, $\displaystyle\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + \text{constant}\,.$

(i) Let $\displaystyle I = \int_0^{\frac{1}{2}\pi} \frac{\cos x}{1 + \sin^2 x}\,\mathrm{d}x\,.$

(a) Evaluate I.

(b) Use the substitution $t = \tan\frac{1}{2}x$ to show that $\displaystyle\int_0^1 \frac{1 - t^2}{1 + 6t^2 + t^4}\,\mathrm{d}t = \tfrac{1}{2}I\,.$

(ii) Evaluate $\displaystyle\int_0^1 \frac{1 - t^2}{1 + 14t^2 + t^4}\,\mathrm{d}t\,.$