

STEP Support Programme

STEP II Pure Questions

- 1 (i) By writing $y = u(1 + x^2)^{\frac{1}{2}}$, where u is a function of x , find the solution of the equation

$$\frac{1}{y} \frac{dy}{dx} = xy + \frac{x}{1 + x^2}$$

for which $y = 1$ when $x = 0$.

- (ii) Find the solution of the equation

$$\frac{1}{y} \frac{dy}{dx} = x^2 y + \frac{x^2}{1 + x^3}$$

for which $y = 1$ when $x = 0$.

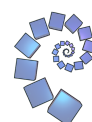
- (iii) Give, without proof, a conjecture for the solution of the equation

$$\frac{1}{y} \frac{dy}{dx} = x^{n-1} y + \frac{x^{n-1}}{1 + x^n}$$

for which $y = 1$ when $x = 0$, where n is an integer greater than 1.

- 2 A curve has equation $y = 2x^3 - bx^2 + cx$. It has a maximum point at (p, m) and a minimum point at (q, n) where $p > 0$ and $n > 0$. Let R be the region enclosed by the curve, the line $x = p$ and the line $y = n$.

- (i) Express b and c in terms of p and q .
- (ii) Sketch the curve. Mark on your sketch the point of inflection and shade the region R . Describe the symmetry of the curve.
- (iii) Show that $m - n = (q - p)^3$.
- (iv) Show that the area of R is $\frac{1}{2}(q - p)^4$.



- 3 (i) By considering the positions of its turning points, show that the curve with equation

$$y = x^3 - 3qx - q(1 + q),$$

where $q > 0$ and $q \neq 1$, crosses the x -axis once only.

- (ii) Given that x satisfies the cubic equation

$$x^3 - 3qx - q(1 + q) = 0,$$

and that

$$x = u + q/u,$$

obtain a quadratic equation satisfied by u^3 . Hence find the real root of the cubic equation in the case $q > 0$, $q \neq 1$.

- (iii) The quadratic equation

$$t^2 - pt + q = 0$$

has roots α and β . Show that

$$\alpha^3 + \beta^3 = p^3 - 3qp.$$

It is given that one of these roots is the square of the other. By considering the expression $(\alpha^2 - \beta)(\beta^2 - \alpha)$, find a relationship between p and q . Given further that $q > 0$, $q \neq 1$ and p is real, determine the value of p in terms of q .

- 4 By writing $x = a \tan \theta$, show that, for $a \neq 0$, $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + \text{constant}$.

(i) Let $I = \int_0^{\frac{1}{2}\pi} \frac{\cos x}{1 + \sin^2 x} dx$.

(a) Evaluate I .

(b) Use the substitution $t = \tan \frac{1}{2}x$ to show that $\int_0^1 \frac{1 - t^2}{1 + 6t^2 + t^4} dt = \frac{1}{2}I$.

(ii) Evaluate $\int_0^1 \frac{1 - t^2}{1 + 14t^2 + t^4} dt$.

