STEP Support Programme

Mechanics Questions: Solutions

2012 S1 Q11

1 First thing to notice is that if $\alpha = \arctan \frac{7}{24}$ then $\tan\alpha = \frac{7}{24}$ and you can use a right-angled triangle to show that $\sin\alpha = \frac{7}{25}$ and $\cos\alpha = \frac{24}{25}$.

Similarly, if $\tan\beta = \frac{4}{3}$ then $\sin\beta = \frac{4}{5}$ and $\cos\beta = \frac{3}{5}$.

(i) Let the tension in the rope be T. Particles A and B are on the point of moving, so the friction on each will be given by $F_r = \mu R$. Resolving parallel to the slopes and equating forces gives us:

$$\begin{aligned} A: &\quad T = 5mg\sin\alpha + 5mg\mu\cos\alpha \\ B: &\quad T = 3mg\sin\beta + 3mg\mu\cos\beta \\ P: &\quad 2T = Mg \end{aligned}$$

Equating the values of T from the first two equations, and substituting for $\sin\alpha$ etc. gives us:

$$\begin{aligned} 5\cancel{mg}\left(\frac{7}{25} + \mu \times \frac{24}{25}\right) &= 3\cancel{mg}\left(\frac{4}{5} + \mu \times \frac{3}{5}\right) \\ 7 + 24\mu &= 12 + 9\mu \\ \mu &= \tfrac{1}{3} \end{aligned}$$

This gives $T = 5mg\left(\frac{7}{25} + \frac{8}{25}\right) = 3mg$ and hence $Mg = 6mg \implies M = 6m$.

(ii) In this case P will be moving downwards. A, B and P can have different accelerations, but they will be connected by $a_A + a_B = 2a_P$.

We have:

$$\begin{aligned} A: &\quad 5ma_A = T - 5mg\sin\alpha - 5mg\mu\cos\alpha \\ B: &\quad 3ma_B = T - 3mg\sin\beta - 3mg\mu\cos\beta \\ P: &\quad Ma_P = Mg - 2T \end{aligned}$$

Using the values of μ and $\sin\alpha$ etc. with $M = 9m$ gives us:

$$\begin{aligned} A: &\quad 5ma_A = T - 3mg \\ B: &\quad 3ma_B = T - 3mg \\ P: &\quad 9ma_P = 9mg - 2T \end{aligned}$$

Equating the first two equations gives us $5ma_A = 3ma_B$. We can add the three equations together to eliminate T and get $9ma_P + 5ma_A + 3ma_B = 3mg$. We now have three equations relating the three accelerations:

$$\begin{aligned} a_A + a_B &= 2a_P \\ 5a_A &= 3a_B \\ 9a_P &= 3g - 5a_A - 3a_B \end{aligned}$$

Solving the simultaneous equations gives us $a_A = \frac{3}{22}g$, $a_B = \frac{5}{22}g$ and $a_P = \frac{4}{22}g$.

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2 The key thing here is a system of labelling the velocities which is easy to follow. I have gone for u_1, u_2, u_3 for the initial velocities, v_1, v_2, v_3 for the velocities after the first collision, w_1, w_2, w_3 for the velocities after the second collision and y_1, y_2, y_3 for the velocities after the third collision.

The starting values are $u_1 = u$, $u_2 = 0$ and $u_3 = 0$. All the masses are the same so we will call them all m.

(i) For the first collision we have:

$$\begin{aligned} \text{Conservation of momentum:} \quad & mu = mv_1 + mv_2 \implies u = v_1 + v_2 \\ \text{Law of restitution:} \quad & eu = v_2 - v_1 \end{aligned}$$

Solving these simultaneously gives:

$$\begin{aligned} v_1 &= \tfrac{1}{2}u(1-e) \\ v_2 &= \tfrac{1}{2}u(1+e)\,. \end{aligned}$$

Note that $v_2 > v_1$ which is what we would expect (particle 1 cannot pass through particle 2).

For the second collision (which will be between particles 2 and 3) we have:

$$\begin{aligned} \text{Conservation of momentum:} \quad & mv_2 = mw_2 + mw_3 \implies v_2 = w_2 + w_3 \\ \text{Law of restitution:} \quad & ev_2 = w_3 - w_2 \end{aligned}$$

Solving these simultaneously gives:

$$\begin{aligned} w_2 &= \tfrac{1}{2}v_2(1-e) \\ w_3 &= \tfrac{1}{2}v_2(1+e) \end{aligned}$$

and we have $w_1 = v_1$.

Substituting $v_2 = \frac{1}{2}u(1+e)$ gives the velocities after the second collision as:

$$\begin{aligned} w_1 &= \tfrac{1}{2}u(1-e) \\ w_2 &= \tfrac{1}{2}(1-e) \times \tfrac{1}{2}u(1+e) = \tfrac{1}{4}u(1-e^2) \\ w_3 &= \tfrac{1}{2}(1+e) \times \tfrac{1}{2}u(1+e) = \tfrac{1}{4}u(1+e)^2 \end{aligned}$$

From above, we have $w_3 > w_2$, as makes sense. For a third collision we need to have $w_1 > w_2$. Consider $w_1 - w_2$:

$$\begin{aligned} w_1 - w_2 &= \tfrac{1}{2}u(1-e) - \tfrac{1}{4}u(1-e^2) \\ &= \tfrac{1}{4}u\left(2(1-e) - (1-e^2)\right) \\ &= \tfrac{1}{4}u\left(1 - 2e + e^2\right) \\ &= \tfrac{1}{4}u(1-e)^2 \end{aligned}$$

and since $e < 1$ we know that $w_1 - w_2 > 0$ and hence $w_1 > w_2$ and there will be another collision for all values of e where $0 < e < 1$.

(ii) For the third collision we have:

$$\begin{aligned} \text{Conservation of momentum:} \quad & mw_1 + mw_2 = my_1 + my_2 \implies w_1 + w_2 = y_1 + y_2 \\ \text{Law of restitution:} \quad & e(w_1 - w_2) = y_2 - y_1 \end{aligned}$$

Pausing to think for a moment, we want to show that there will be a fourth collision which means we want $y_2 > w_3$. Hence we don't actually need to find y_1!

Solving the equations for y_2 gives:

$$\begin{aligned} y_2 &= \tfrac{1}{2}(w_1 + w_2 + ew_1 - ew_2) \\ &= \tfrac{1}{2}\left(w_1(1+e) + w_2(1-e)\right) \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}u(1-e)(1+e) + \tfrac{1}{4}u(1-e^2)(1-e)\right) \end{aligned}$$

We have a fourth collision iff $y_2 - w_3 > 0$, so we want:

$$\begin{aligned} \tfrac{1}{8}\left(2u(1-e)(1+e) + u(1-e^2)(1-e)\right) - \tfrac{1}{4}u(1+e)^2 &> 0 \\ \tfrac{1}{8}u(1+e)\left(2(1-e) + (1-e)^2 - 2(1+e)\right) &> 0 \\ \tfrac{1}{8}u(1+e)\left(e^2 - 6e + 1\right) &> 0 \end{aligned}$$

u and $1+e$ are positive, so we need $e^2 - 6e + 1 > 0$. Solving $e^2 - 6e + 1 = 0$ gives the solutions $e = \dfrac{6 \pm \sqrt{32}}{2} = 3 \pm 8$ and — remembering that $0 < e < 1$ — we can conclude that there will be a fourth collision iff $0 < e < 3 - \sqrt{8}$.