Answer Sheet 3

1. $\frac{2x}{2x^2-5x+3}+\frac{13x}{2x^2+x+3}=6$. This is a quartic equation in x . Observing $2x^2+3$ in both fractions we divide top and bottom of the fraction by x and put $2x+\frac{3}{x}=y$ then $\frac{2}{y-5}+\frac{13}{y+1}=6$. We can now clear fractions and solve the quadratic equation $2y^2-13y+11=0$ or we could observe another symmetry, substitute $z=y-2$ and simplify the expression to give $2z^2-5z-7=0$. The solutions are $z=\frac{7}{2}$ or -1 giving $y=\frac{11}{2}$ or 1 then we have $2x+\frac{3}{x}=\frac{11}{2}$ and $2x+\frac{3}{x}=1$. We then have to solve the equations $2x^2-x+3=0$ and $4x^2-11x+6=0$. The solutions are $x=\frac{1\pm\sqrt{-23}}{4}$ and $x=\frac{3}{4}$ or 2.

2. $x^4-8x^3+17x^2-8x+1=0$. This is another quartic and, noting the symmetry in the coefficients, we use the substitution $t=x+\frac{1}{x}$ and $t^2=x^2+2+\frac{1}{x^2}$ to turn it into a quadratic equation. Dividing by x^2: $x^2-8x+17-\frac{8}{x}+\frac{1}{x^2}=0$ $t^2-2-8t+17=0$ so $t^2-8t+15=0$ and the solutions are $t=3$ or $t=5$. If $x^2-tx+1=0$ then $x=\frac{t\pm\sqrt{t^2-4}}{2}$ and so the solutions are $x=\frac{3\pm\sqrt{5}}{2}$ and $x=\frac{5\pm\sqrt{21}}{2}$.

3. $(x-4)(x-5)(x-6)(x-7)=1680$. The symmetry here is about $(x-5.5)$ so we write $t=x-5.5$ and the equation becomes: $(t+\frac{3}{2})(t+\frac{1}{2})(t-\frac{1}{2})(t-\frac{3}{2})=1680=40\times 42$ $(t^2-\frac{9}{4})(t^2-\frac{1}{4})=1680$ Another symmetry helps here, we substitute $t^2=s+\frac{5}{4}$ and the equation becomes $(s-1)(s+1)=1680$ so $s^2=1681$ and $s=\pm 41$. Now going back to the original equation

$$s=t^2-\frac{5}{4}=x^2-11x+\frac{121}{4}-\frac{5}{4}=x^2-11x+29$$

so we have to solve the two quadratic equations $x^2-11x+29=\pm 41$. The solutions are $x=-1$ or 12 or $\frac{11\pm i\sqrt{159}}{2}$.

4. $(8x+7)^2(4x+3)(x+1)=\frac{9}{2}$. To make this simpler we try the substitution $t=8x+7$ but we have to multiply the second bracket by 2 and the third by 8 so we get: $(8x+7)^2(8x+6)(8x+8)=\frac{9}{2}\times 16$ giving $t^2(t-1)(t+1)=72$ that is $t^4-t^2-72=(t^2-9)(t^2+8)=0$ so $t=\pm 3$ or $t=\pm 2i\sqrt{2}$. $x=\frac{1}{8}(t-7)$ so the solutions are: $x=-\frac{1}{2},-\frac{5}{4},\frac{2i\sqrt{2}-7}{8},\frac{-2i\sqrt{2}-7}{8}$.

5. In this case make the substitution $y=x+4$, which transforms the equation into

$$(y-1)^4+(y+1)^4=20.$$

This is perhaps the simplest form of the equation under a linear transformation as the two factors are now at least 'similar'.

If I expand these brackets then the coefficients will match but with some opposing signs, so much cancellation will occur:

$$(y^4-4y^3+6y^2-y+1)+(y^4+4y^3+6y^2+y+1)=20.$$

So, the odd factors cancel to give

$$2y^4+12y^2-18=0$$

This then becomes the simple equation

$$y^4 + 6y^2 - 9 = 0\,,$$

which is a quadratic equation in y^2 with solutions

$$y^2 = \frac{-6 \pm \sqrt{36+36}}{2} = -3(1 \pm \sqrt{2}).$$

One of these solutions is positive; taking the square root gives two real values for y as

$$y = \pm\sqrt{3\left(\sqrt{2}-1\right)}$$

Therefore two real solutions to the equation are

$$x = -4 \pm \sqrt{3\left(\sqrt{2}-1\right)}$$