Answer Sheet 2

1. If $a^2 + b^2 = c^2$ then
$$\frac{1}{b^2c^2} + \frac{1}{a^2c^2} = \frac{1}{a^2b^2}.$$
So every Pythgorean triple gives rise to a solution to our problem. The smallest solution of this type arises from $a = 3, b = 4, c = 5$ and gives
$$\frac{1}{20^2} + \frac{1}{15^2} = \frac{1}{12^2}.$$
Students may not be familiar with Pythagorean Triples - this would be a good area for them to read up on if they're interested, and there's lots of good stuff on NRICH and elsewhere on the web.

2. Let S be the mininum value, $S = 1/u + 1/v = (v + u)/(uv)$. Since $u + v = 5$ then $S = 5/uv$. The maximum value of uv gives a mininum value of S. Given $u + v = 5$ then $v = 5 - u$. Let $f(u)$ give the value of uv as u and v change. Then $f(u) = u(5 - u) = -u^2 + 5u$. This is a quadratic function and the vertex (u, M) of the graph of $f(u)$ is at $(2.5, 6.25)$. It means the maximum value occurs at $M = 6.25$ when $u = 2.5$ and $v = 5 - u = 2.5$. So the mininum value $S = 1/2.5 + 1/2.5 = 0.8$.

3. Square both sides. Multiply throughout by x. Rearrange to form the quadratic inequality:
$$x^2 - 14x + 1 < 0.$$
Use the quadratic formula to solve this inequality. From the graph of $y = x^2 - 14x + 1$ we see that the solution is $7 - 4\sqrt{3} < x < 7 + 4\sqrt{3}$

4. $(1 - x)(1 - y) > 0$
$1 - x - y + xy > 0$
$1 + xy > x + y$
This is equivalent to $x + y < 1 + xy$

5.
$$\text{Square of mean } = \frac{(p+q)^2}{4} = \frac{p^2 + 2pq + q^2}{4}$$
$$\text{Mean of squares } = \frac{p^2 + q^2}{2}$$
.
$$\frac{p^2 + q^2}{2} - \frac{p^2 + 2pq + q^2}{4} = \frac{p^2 - 2pq + q^2}{4} = \frac{(p-q)^2}{4} \geq 0.$$
.
Note that this difference, $\frac{(p-q)^2}{4}$, is zero if $p = q$ and positive for all other choices of p and q. So if the numbers are equal then the mean of the squares is equal to the square of the mean. Otherwise the mean of the squares is greater than the square of the mean.

6. The key here is that x has to be the integer part because the 'continued fraction' part of the expression gives a value less than one.

   As y and z are positive integers (whole numbers), $y + 1/z > 1$ and $1/(y + 1/z) < 1$ so we know that this must equal $3/7$ and $x = 1$.

   Hence $y + 1/z = 7/3$. Again y has to be the integer part of $7/3$ so $y = 2$ and $z = 3$.