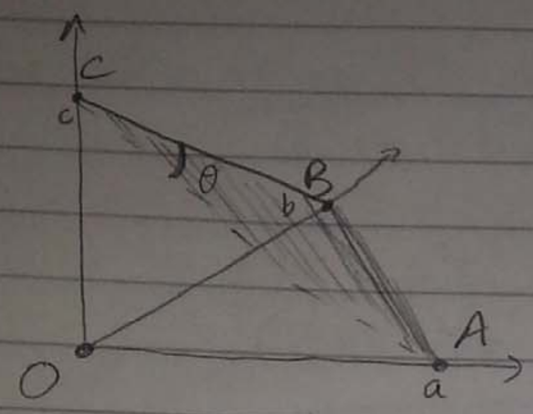
8.



(i) volume of tetrahedron OABC $= \frac{1}{3} \times \text{base area} \times \text{height}$

$$= \frac{1}{3} \times \left(\frac{ab}{2}\right) \times c$$

$$= \frac{1}{6} abc$$

(ii) $|AB| = \sqrt{a^2+b^2}$

$|AC| = \sqrt{a^2+c^2}$

$|BC| = \sqrt{b^2+c^2}$

By the cosine rule,

$$|AB|^2 = |AC|^2 + |BC|^2 - 2|AC||BC|\cos\theta$$

$$\Rightarrow a^2+b^2 = a^2+c^2+b^2+c^2 - 2\sqrt{(a^2+c^2)(b^2+c^2)}\cos\theta$$

$$\Rightarrow 2\sqrt{(a^2+c^2)(b^2+c^2)}\cos\theta = 2c^2$$

$$\Rightarrow \cos\theta = \frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$$ as required.

To find the area of triangle ABC, first work out $\sin\theta$, then use Area of triangle $= \frac{1}{2}|AC||BC|\sin\theta$.

$$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \left(\frac{c^4}{(a^2+c^2)(b^2+c^2)}\right)$$

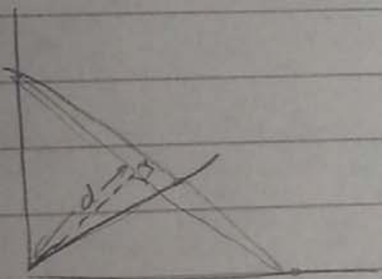
$$\sin^2\theta = \frac{(a^2+c^2)(b^2+c^2) - c^4}{(a^2+b^2)(b^2+c^2)}$$

$$= \frac{a^2b^2 + a^2c^2 + b^2c^2}{(a^2+c^2)(b^2+c^2)}$$

$$\therefore \sin\theta = \pm\sqrt{\frac{a^2b^2 + a^2c^2 + b^2c^2}{(a^2+c^2)(b^2+c^2)}}$$ (but $0 < \theta < \pi$ so $\sin\theta > 0$)

So area of triangle ABC $= \frac{1}{2}\sqrt{a^2+c^2}\sqrt{b^2+c^2}\,\sin\theta$

$$= \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}.$$



Let d be the perpendicular distance from the origin to triangle ABC.

Then the volume of the tetrahedron can be given as

$$V = \frac{\text{area of ABC} \times d}{3}$$

$$= \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}\,\frac{d}{3}$$

But from (i) we know $V = \frac{1}{6}abc$

Equating volumes gives

$$abc = \sqrt{a^2b^2 + a^2c^2 + b^2c^2}\; d$$

$$\Rightarrow \quad d^2 = \frac{a^2b^2c^2}{a^2b^2 + a^2c^2 + b^2c^2}$$

$$\Rightarrow \quad \frac{1}{d^2} = \frac{a^2b^2 + a^2c^2 + b^2c^2}{a^2b^2c^2} = \frac{1}{c^2} + \frac{1}{b^2} + \frac{1}{a^2}$$