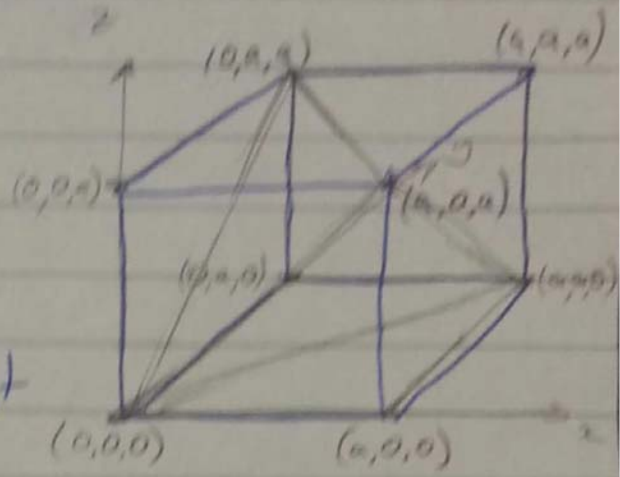
Consider the cube with vertices
$\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\a\\0\end{pmatrix}\begin{pmatrix}0\\0\\a\end{pmatrix}\begin{pmatrix}a\\0\\0\end{pmatrix}$ $\begin{pmatrix}a\\a\\0\end{pmatrix}\begin{pmatrix}0\\a\\a\end{pmatrix}\begin{pmatrix}a\\0\\a\end{pmatrix}\begin{pmatrix}a\\a\\a\end{pmatrix}$



Choosing the origin as one vertex, the other three non-adjacent vertices to $\begin{pmatrix}0\\0\\0\end{pmatrix}$ are

$A: \begin{pmatrix}a\\a\\0\end{pmatrix}$, $B: \begin{pmatrix}a\\0\\a\end{pmatrix}$, $C: \begin{pmatrix}0\\a\\a\end{pmatrix}$

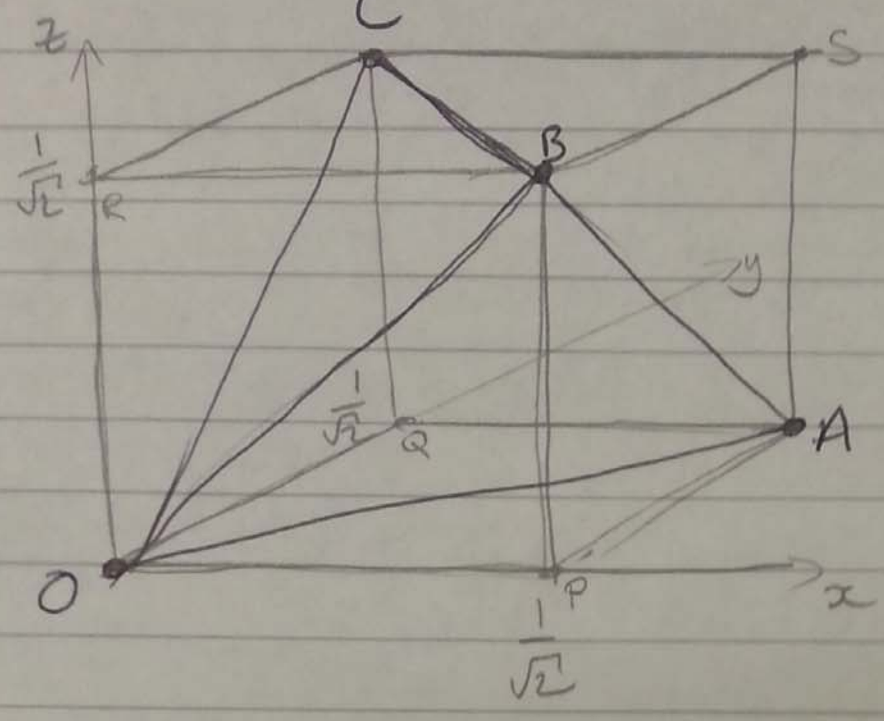
Thus there are six edges joining these four vertices, given by the vectors:

$$\left.\begin{array}{ll} OA & \begin{pmatrix}a\\a\\0\end{pmatrix} \\ OB & \begin{pmatrix}a\\0\\a\end{pmatrix} \\ OC & \begin{pmatrix}0\\a\\a\end{pmatrix} \\ AB & \begin{pmatrix}0\\-a\\a\end{pmatrix} \\ AC & \begin{pmatrix}-a\\0\\a\end{pmatrix} \\ BC & \begin{pmatrix}-a\\a\\0\end{pmatrix} \end{array}\right\}$$

These vectors all have equal magnitude $\sqrt{2}a$, so each triangular face is equilateral. Therefore the tetrahedron is regular.

To work out the volume of the tetrahedron with unit side length, I can subtract the volume of the four equal right ~~pyramids~~ tetrahedra from the volume of the appropriate cube.

A cube with side length a contains a tetrahedron with side length $\sqrt{2}a$, therefore I need to consider the cube with side length $\frac{1}{\sqrt{2}}$ in order to calculate the volume of the unit tetrahedron.



Let the other four vertices of the cube be

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \quad R = \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \quad S = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

The volume of the tetrahedron OABC is equal to the volume of the cube with the volume of the tetrahedra OABP, OBCR, OACQ and ABCS subtracted.

Each tetrahedron has volume

$$\frac{1}{3} \times \text{base area} \times \text{height}$$

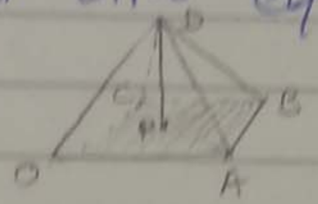
$$= \frac{1}{3} \times \left(\frac{1}{2} \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \right) \times \frac{1}{\sqrt{2}} = \frac{1}{12\sqrt{2}}$$

The cube has volume $\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{2\sqrt{2}}$

$\therefore$ The tetrahedron OABC has volume

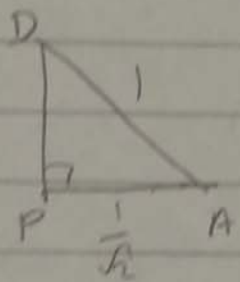
$$\frac{1}{2\sqrt{2}} - 4\left(\frac{1}{12\sqrt{2}}\right) = \frac{1}{6\sqrt{2}}$$

Consider half of the unit octahedron - a square-based pyramid with equilateral triangular faces:



Label the pyramid's vertices OABC around the square, and D for the apex.
Let P be the point perpendicularly below D on the square

AP is one side of an isosceles right-angled triangle, so $|AP| = \frac{1}{\sqrt{2}}$.



Triangle PAD is the same triangle, so $|PD| = \frac{1}{\sqrt{2}}$.

So the volume of the pyramid is

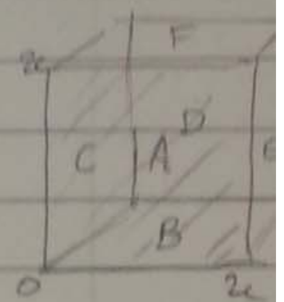
$$\frac{1}{3} \times \text{base area} \times \text{height}$$

$$= \frac{1}{3} \times 1 \times \frac{1}{\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

Thus the volume of the octahedron is $\frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$.

Consider a cube with vertices

$$\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}2c\\0\\0\end{pmatrix}\begin{pmatrix}0\\2c\\0\end{pmatrix}\begin{pmatrix}0\\0\\2c\end{pmatrix}\begin{pmatrix}2c\\2c\\0\end{pmatrix}\begin{pmatrix}2c\\0\\2c\end{pmatrix}\begin{pmatrix}0\\2c\\2c\end{pmatrix}\begin{pmatrix}2c\\2c\\2c\end{pmatrix}$$



The midpoints of the faces are then

$$\underset{A}{\begin{pmatrix}c\\0\\c\end{pmatrix}}\underset{B}{\begin{pmatrix}c\\c\\0\end{pmatrix}}\underset{C}{\begin{pmatrix}0\\c\\c\end{pmatrix}}\underset{D}{\begin{pmatrix}c\\2c\\c\end{pmatrix}}\underset{E}{\begin{pmatrix}2c\\c\\c\end{pmatrix}}\underset{F}{\begin{pmatrix}c\\c\\2c\end{pmatrix}}$$

A & D opposite
C & E opposite
B & F opposite

Joining these midpoints:

$\overrightarrow{AB}\begin{pmatrix}0\\c\\-c\end{pmatrix}$ $\quad\overrightarrow{BC}\begin{pmatrix}-c\\0\\c\end{pmatrix}$ $\quad\overrightarrow{CF}\begin{pmatrix}c\\0\\c\end{pmatrix}$

$\overrightarrow{AC}\begin{pmatrix}-c\\c\\0\end{pmatrix}$ $\quad\overrightarrow{BD}\begin{pmatrix}0\\c\\c\end{pmatrix}$ $\quad\overrightarrow{DE}\begin{pmatrix}c\\-c\\0\end{pmatrix}$

$\overrightarrow{AE}\begin{pmatrix}c\\c\\0\end{pmatrix}$ $\quad\overrightarrow{BE}\begin{pmatrix}c\\0\\c\end{pmatrix}$ $\quad\overrightarrow{DF}\begin{pmatrix}0\\-c\\c\end{pmatrix}$

$\overrightarrow{AF}\begin{pmatrix}0\\c\\c\end{pmatrix}$ $\quad\overrightarrow{CD}\begin{pmatrix}c\\c\\0\end{pmatrix}$ $\quad\overrightarrow{EF}\begin{pmatrix}-c\\0\\c\end{pmatrix}$

The twelve vectors have the same magnitude, therefore the faces are all equilateral triangles. So the solid formed by joining midpoints of the faces is a regular octahedron.

From part (i): volume of cube $\frac{1}{2\sqrt{2}}$
volume of tetrahedron $\frac{1}{6\sqrt{2}}$
$\therefore$ volume of tetrahedron is $\frac{1}{3}$ volume of cube

From part (ii): volume of cube $(\sqrt{2})^3 = 2\sqrt{2}$
volume of octahedron $= \sqrt{2}/3$
$\therefore$ volume of octahedron is $\frac{1}{6}$ volume of cube.

So the volume of the octahedron is half the volume of the tetrahedron.