Wait to show $y = e^x$ is a solution to (\#).

Suppose $y = e^x$. Then $\frac{dy}{dx} = \frac{d^2y}{dx^2} = e^x$.

LHS of (\#) = $(x-1)e^x - xe^x + e^x$

= 0 = RHS of (\#)

so $y = e^x$ satisfies (\#).

Let $y = u e^x$ for some $u = f(x)$.

Then $\frac{dy}{dx} = u e^x + e^x \frac{du}{dx}$ (using product rule)

$\Rightarrow \frac{d^2y}{dx^2} = u e^x + 2e^x \frac{du}{dx} + e^x \frac{d^2u}{dx^2}$ (product rule again).

Substituting into (\#),

$(x-1) \left[ u e^x + 2e^x \frac{du}{dx} + e^x \frac{d^2u}{dx^2} \right] - x \left[ u e^x + e^x \frac{du}{dx} \right] + u e^x = 0.$

Simplifying,

$(x-2) e^x \frac{du}{dx} + (x-1) e^x \frac{d^2u}{dx^2} = 0.$

Now $e^x \neq 0$ for $x$ real so this becomes

$(x-1) \frac{d^2u}{dx^2} + (x-2) \frac{du}{dx} = 0.$ (\#\#).

Write $v = \frac{du}{dx}$.

Then (\#\#) becomes

$(x-1) \frac{dv}{dx} + (x-2) v = 0.$

$\Rightarrow (x-1) \frac{dv}{dx} = (2-x) v$.

Integrating,

$\Rightarrow \frac{1}{v} \frac{dv}{dx} = \frac{2-x}{x-1}$

$\Rightarrow \int \frac{1}{v} dv = \int \frac{2-x}{x-1} dx.$
\[ \ln |v| = \int -1 + \frac{1}{x-1} \, dx \quad \text{← have written } \frac{2-x}{x-1} = 1 + \frac{1}{x-1} \]

\[ = -x + \ln |x-1| + C \quad \text{This is a really useful trick for integrating fractions.} \]

\[ c = \text{const.} \]

Exponentiate to get

\[ N = |k| e^{-x} |x-1| \quad k = e^c. \]

so take \[ \nu = k e^{-x} (x-1). \quad \text{← Don't forget to continue...} \]

\[ \Rightarrow \frac{du}{dx} = ke^{-x}(x-1). \]

Integrating,

\[ u = k \int e^{-x} - e^{-x} \, dx. \]

\[ = k \left[ -xe^{-x} - e^{-x} + C \right] \]

\[ = -kxe^{-x} + l \]

\[ l = \text{const.} \quad \text{Don't forget to continue, still not done yet!} \]

Substituting into \[ y = u e^{x} \] gives

\[ y = -kx + le^{x}. \]

Set \( A = -k, B = l \) to give

\[ y = Ax + Be^{x} \] a solution to (4), as required.