2010 STEP I Question 5.

The Binomial Theorem tells us that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n \quad (*).$$

This is the key to the whole question.

(i) Setting $x=1$ gives $2^n = (1+1)^n$

$$= \binom{n}{0} + \binom{n}{1}\cdot 1 + \binom{n}{2}\cdot 1^2 + \dots + \binom{n}{n}\cdot 1^n$$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} \quad \text{as required.}$$

Seems sensible; the $n\cdot 2^{n-1}$ is a pretty good hint.

(ii) Differentiate $(*)$ to give

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \dots + n\binom{n}{n}x^{n-1}. \quad (**).$$

Now set $x=1$ to give

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n} \quad \text{as required.}$$

Again, seems sensible given the $\frac{2^{n+1}}{n+1}$ and part (ii).

(iii) Integrate $(*)$ to give

$$\frac{(1+x)^{n+1}}{n+1} + \text{const} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \dots + \binom{n}{n}\frac{x^{n+1}}{n+1}.$$

Careful!

To figure out the constant, require equation to hold when $x=0$:

Then RHS $=0$.

LHS $= \frac{1}{n+1} + \text{const}$. So const $= \frac{-1}{n+1}$.

$$\text{So } \frac{(1+x)^{n+1}-1}{n+1} = \binom{n}{0}x + \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \dots + \binom{n}{n}\frac{x^{n+1}}{n+1}.$$

Now set $x=1$ to give

$$\frac{1}{n+1}(2^{n+1}-1) = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{1}{n+1}\binom{n}{n} \quad \text{as required.}$$

(iv) Multipy (**) throughout by x to give ← Unsurprisingly, this is the hardest bit. It would definitely be useful to spot the similarity to the identity from part (ii).

$$nx(1+x)^{n-1} = \binom{n}{1}x + 2\binom{n}{2}x^2 + 3\binom{n}{3}x^3 + \dots + n\binom{n}{n}x^n.$$

Differentiating,

$$n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2} = \binom{n}{1} + 2^2\binom{n}{2}x + 3^2\binom{n}{3}x^2 + \dots + n^2\binom{n}{n}x^{n-1}.$$

Now set $x=1$:

$$n2^{n-1} + n(n-1)\cdot 2^{n-2} = \binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \dots + n^2\binom{n}{n}.$$

Simplifying LHS: $n2^{n-1} + n(n-1)\cdot 2^{n-2} = 2^{n-2}(2n + n(n-1))$

$$= 2^{n-2}(n^2+n)$$

$$= n(n+1)2^{n-2}.$$

so $n(n+1)2^{n-2} = \binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \dots + n^2\binom{n}{n}$, as required.