Using the substitution $x = \frac{1}{t^2-1}$, for $t > 1$, (Note that this is a valid substitution because we can obtain any value of $x > 0$ using some $t > 1$).

This sort of thing is worth checking (even more so if they haven't given you the substitution)

For $x > 0$,

$$\frac{dx}{dt} = \frac{-2t}{(t^2-1)^2}.$$

$$\int \frac{1}{\sqrt{x(x+1)}}\,dx = \int \frac{1}{\sqrt{\left(\frac{1}{t^2-1}\right)\cdot\left(1+\frac{1}{t^2-1}\right)}} \cdot \frac{dx}{dt}\,dt$$

$$= -2\int \sqrt{\frac{(t^2-1)^2}{t^2-1+1}} \cdot \frac{t}{(t^2-1)^2}\,dt$$

Yuck! Careful with the algebra...

$$= -2\int \sqrt{\frac{(t^2-1)^2}{t^2}} \cdot \frac{t}{(t^2-1)^2}\,dt$$

$$= -2\int \frac{t^2-1}{t} \cdot \frac{t}{(t^2-1)^2}\,dt$$

$$= -2\int \frac{1}{t^2-1}\,dt$$

$$= -\ln\left|\frac{t-1}{t+1}\right| + c \qquad c = \text{const.}$$

Now
$$(\sqrt{x}+\sqrt{x+1})^2 = x + 2\sqrt{x(x+1)} + x + 1$$

$$= \frac{2}{t^2-1} + 1 + 2\sqrt{\frac{1}{t^2-1}\left(\frac{1}{t^2-1}+1\right)}$$

$$= \frac{t^2+1}{t^2-1} + 2\sqrt{\frac{t^2}{(t^2-1)^2}}$$

$$= \frac{t^2+2t+1}{t^2-1}$$

$$= \frac{(t+1)^2}{(t-1)(t+1)}$$

$$= \frac{t+1}{t-1}.$$

so $-\ln\left[(\sqrt{x}+\sqrt{x+1})^2\right] = \ln\left(\frac{t+1}{t-1}\right)$.

so
$$\int \frac{1}{\sqrt{x(x+1)}}\,dx = \ln\left[(\sqrt{x}+\sqrt{x+1})^2\right] + c$$

$$= 2\ln(\sqrt{x}+\sqrt{x+1}) + c.$$

Now want to find volume of revolution V of curve $y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$ about x-axis between $x = \frac{1}{8}$ and $x = \frac{9}{16}$.

$$V = \pi \int_{1/8}^{9/16} y^2 \, dx$$

$$= \pi \int_{1/8}^{9/16} \frac{1}{x} - \frac{2}{\sqrt{x(x+1)}} + \frac{1}{x+1} \, dx.$$

$$= \pi \Big[\ln x - 4 \ln(\sqrt{x} + \sqrt{x+1}) + \ln(x+1) \Big]_{1/8}^{9/16}.$$

$$= \pi \Big[(\ln 9/16 - 4 \ln 2 + \ln 25/16) - (\ln 1/8 - 4 \ln(\sqrt{2}) - \ln(9/8)) \Big]$$

You can do some of the calculations on scrap paper. (I didn't do this all in my head!).

$$= \pi \Big[\ln 9/16 + \ln 25/16 - \ln 16 - \ln 1/8 + \ln 4 - \ln 9/8 \Big]$$

$$= \pi \ln \left[\frac{9/16 \cdot 25/16 \cdot 4}{16 \cdot 1/8 \cdot 9/8} \right]$$

$$= \pi \ln \left[\frac{\cancel{9} \cdot 25 \cdot \cancel{4 \cdot 8 \cdot 8}}{16 \cdot \cancel{16 \cdot 16} \cdot \cancel{9}} \right]$$

$$= \pi \ln \left(\frac{25}{16} \right)$$

$$= \pi \ln \left[\left(\frac{5}{4} \right)^2 \right]$$

$$= 2\pi \ln \left(\frac{5}{4} \right).$$