2010 STEP II Question 3

(i) We have $F_n = a\lambda^n + b\mu^n$.

So $F_0 = a + b = 0$. (1)

$F_1 = a\lambda + b\mu = 1$ (2)

$F_2 = a\lambda^2 + b\mu^2 = 1$ (3)

$F_3 = a\lambda^3 + b\mu^3 = 2$. (4).

(1) $\Rightarrow$ $b = -a$.

so (2) becomes $a(\lambda-\mu) = 1$

(3) becomes $a(\lambda^2-\mu^2) = 1$

(4) becomes $a(\lambda^3-\mu^3) = 2$.

Now $(\lambda^3-\mu^3) = (\lambda-\mu)(\lambda^2+\mu\lambda+\mu^2)$. ← This kind of factorisation pops up quite a lot.

So (4) ÷ (2) gives $\dfrac{a(\lambda^3-\mu^3)}{a(\lambda-\mu)} = \dfrac{2}{1}$

$\Rightarrow$ $\lambda^2+\mu\lambda+\mu^2 = 2$. (5)

Now if we do (3) ÷ (2) we get $\dfrac{a(\lambda^2-\mu^2)}{a(\lambda-\mu)} = 1$ ← This bit is fiddly. There are other ways to do it. Be careful of going round in circles.

$\Rightarrow$ $\lambda+\mu = 1$.

$\Rightarrow$ $(\lambda+\mu)^2 = 1$.

$\Rightarrow$ $\lambda^2+2\mu\lambda+\mu^2 = 1$. (6).

Now (6) − (5) $\Rightarrow$ $\mu\lambda = -1$.

with $\lambda+\mu = 1$, this gives $\lambda - \frac{1}{\lambda} = 1$

$\Rightarrow$ $\lambda^2-\lambda-1 = 0$.

$\Rightarrow$ $\lambda = \dfrac{1\pm\sqrt{5}}{2}$.

(1) $\Rightarrow$ $a(\lambda-(1-\lambda)) = 1$

$\Rightarrow$ $a(2\lambda-1) = 1$.

$\Rightarrow$ $a(1\pm\sqrt{5}-1) = 1$.

$\Rightarrow$ $a(\pm\sqrt{5}) = 1$.

a is positive so take $\lambda = \dfrac{1+\sqrt{5}}{2}$ and $a = \dfrac{1}{\sqrt{5}}$.

So $\mu = \dfrac{1-\sqrt{5}}{2}$ and $b = -\dfrac{1}{\sqrt{5}}$.

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(ii) $F_6 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^6 - \left(\frac{1-\sqrt{5}}{2}\right)^6\right)$

$$= \frac{1}{2^6\sqrt{5}}\underbrace{\left((1+\sqrt{5})^6 - (1-\sqrt{5})^6\right)}$$

Expanding these using Binomial theorem, the even powers of $\sqrt{5}$ will cancel and the odd powers are the same. ← Just a clever thought to save some arithmetic. Expanding out both terms and calculating them separately is also fine.

$$= \frac{1}{2^6\sqrt{5}}\cdot 2(6\sqrt{5} + 20\times 5\sqrt{5} + 6\times 25\sqrt{5})$$

$$= \frac{1}{2^5\sqrt{5}}\cdot 256\sqrt{5}$$

$$= 8.$$

Good to leave in terms of λ and μ at this point to save writing

(iii) $$\sum_{n=0}^{\infty}\frac{F_n}{2^{n+1}} = \frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty}\left(\frac{\lambda^n}{2^{n+1}}\right) - \sum_{n=0}^{\infty}\left(\frac{\mu^n}{2^{n+1}}\right)\right]$$

$$= \frac{1}{2\sqrt{5}}\left[\sum_{n=0}^{\infty}\left(\frac{\lambda}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{\mu}{2}\right)^n\right]$$

$$= \frac{1}{2\sqrt{5}}\left[\frac{1}{1-\lambda/2} - \frac{1}{1-\mu/2}\right]$$ ← using limit of sum of geometric progression.

$$= \frac{1}{2\sqrt{5}}\left[\frac{1}{3/4-\sqrt{5}/4} - \frac{1}{3/4+\frac{\sqrt{5}}{4}}\right]$$

$$= \frac{2}{\sqrt{5}}\left[\frac{1}{3-\sqrt{5}} - \frac{1}{3+\sqrt{5}}\right]$$

$$= \frac{2}{\sqrt{5}}\left[\frac{3+\sqrt{5}}{4} - \frac{3-\sqrt{5}}{4}\right]$$

$$= \frac{4\sqrt{5}}{4\sqrt{5}} = 1.$$