2010 STEP III Q4

i) We have that $\alpha^2 + a\alpha + b = 0$ and $\alpha^2 + c\alpha + d = 0$, so taking the second equation away from the first gives

$$(a-c)\alpha + (b-d) = 0 \quad \Rightarrow \quad \alpha = -\left(\frac{b-d}{a-c}\right).$$

we can divide by $a-c$ since $a \neq c$.

If $(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0$, then we can divide through by $(a-c)^2$:

$$\left(\frac{b-d}{a-c}\right)^2 + a\left(-\frac{b-d}{a-c}\right) + b = 0, \quad \text{so} \quad \alpha = -\left(\frac{b-d}{a-c}\right) \text{ satisfies } x^2 + ax + b = 0.$$

But also

$$\left(\frac{b-d}{a-c}\right)^2 + c\left(-\frac{b-d}{a-c}\right) + d + \cancel{(a-c)}\left(-\frac{\cancel{b-d}}{\cancel{a-c}}\right) + \cancel{(b-d)} = 0$$

$$\Rightarrow \quad \alpha = -\left(\frac{b-d}{a-c}\right) \text{ also satisfies } x^2 + cx + d = 0.$$

So α is a common root of the equations.

It's an 'if and only if' statement, so we need to check both ways.

If, on the other hand, α is a common root of $x^2 + ax + b = 0$ & $x^2 + cx + d = 0$, then as before, $\alpha = -\left(\frac{b-d}{a-c}\right)$, and since α satisfies $x^2 + ax + b = 0$, we have

$$\left(\frac{b-d}{a-c}\right)^2 - a\left(\frac{b-d}{a-c}\right) + b = 0 \quad \Rightarrow \quad (b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0.$$

If we can have $a = c$, then $x^2 + ax + b = 0$ and $x^2 + ax + d = 0$ have a common root if and only if $b = d$, so then the two equations are equal, and so they always have a common root, and

$$(b-d)^2 - a(b-d)(a-c) + b(a-c)^2 = 0 \quad \text{is} \quad 0 = 0,$$ so is trivially true.

So the result still holds for $a = c$.

ii) If α is a common root of $x^2+ax+b=0$ & $x^3+(a+1)x^2+qx+r=0$,

then $\alpha^3+(a+1)\alpha^2+q\alpha+r=0$ ①

$\alpha^2+a\alpha+b=0 \Rightarrow \alpha^3+a\alpha^2+b\alpha=0$. ②

Then ① - ② gives $\alpha^2+(q-b)\alpha+r=0$.

So α satisfies the quadratics $x^2+ax+b=0$ & $x^2+(q-b)x+r=0$, so as in part (i), $\alpha = -\dfrac{b-r}{a-(q-b)} = -\left(\dfrac{b-r}{a+b-q}\right)$.

Now we can use the result from part (i), with $c=q-b$ & $d=r$:

α is a common root of $x^2+ax+b=0$ & $x^3+(a+1)x^2+qx+r=0$ if and only if

$(b-r)^2-a(b-r)(a+b-q)+b(a+b-q)^2=0$ as required.

Now $x^2+\frac{5}{2}x+b=0$, $x^3+\frac{7}{2}x^2+\frac{5}{2}x+\frac{5}{2}=0$,

so $a=\frac{5}{2}$, $q=\frac{5}{2}$, $r=\frac{1}{2}$.

So the equations have at least one common root if and only if

$(b-\frac{1}{2})^2-\frac{5}{2}(b-\frac{1}{2})(\frac{5}{2}+b-\frac{5}{2})+b(\frac{5}{2}+b-\frac{5}{2})^2=0$

$\Leftrightarrow b^3-\frac{5}{2}b(b-\frac{1}{2})+(b-\frac{1}{2})^2=0$

$\Leftrightarrow 4b^3-6b^2+b+1=0$

$\Leftrightarrow (b-1)(4b^2-2b-1)=0$

$\Leftrightarrow \boxed{b=1}, \boxed{b=\frac{1+\sqrt{5}}{4}}$ or $\boxed{b=\frac{1-\sqrt{5}}{4}}$.