2010 STEP III Q2

i) $\cosh a = \frac{1}{2}(e^{a} + e^{-a})$.

$$\int_0^1 \frac{1}{x^2 + 2x\cosh a + 1}\,dx = \int_0^1 \frac{1}{x^2 + (e^a + e^{-a})x + 1}\,dx = \int_0^1 \frac{1}{(x+e^a)(x+e^{-a})}\,dx$$

we can factorise this!

$$= \int_0^1 \frac{A}{x+e^a} + \frac{B}{x+e^{-a}}\,dx \quad , \ A \text{ and } B \text{ some constants.}$$

We have that $A(x+e^{-a}) + B(x+e^{a}) \equiv 1$, so comparing coefficients:

$A + B = 0$, $Ae^{-a} + Be^{a} = 1$

$\Rightarrow A = -B$ $\quad \Rightarrow B(e^{a} - e^{-a}) = 1$

$$\Rightarrow B = \frac{1}{2\sinh a}$$

So our integral is $\displaystyle \frac{1}{2\sinh a}\int_0^1 \frac{1}{x+e^{-a}} - \frac{1}{x+e^{a}}\,dx$

$$= \frac{1}{2\sinh a}\Big[\ln(x+e^{-a}) - \ln(x+e^{a})\Big]_0^1$$

$$= \frac{1}{2\sinh a}\left[\ln\left(\frac{x+e^{-a}}{x+e^{a}}\right)\right]_0^1$$

$$= \frac{1}{2\sinh a}\left(\ln\left(\frac{1+e^{-a}}{1+e^{a}}\right) - \ln\left(\frac{e^{-a}}{e^{a}}\right)\right)$$

$$= \frac{1}{2\sinh a}\left(\ln\left(e^{-a}\cdot\frac{\cancel{1+e^{a}}}{\cancel{1+e^{a}}}\right) - \ln\left(e^{-2a}\right)\right)$$

$$= \frac{1}{2\sinh a}(-a - -2a)$$

$$= \boxed{\frac{a}{2\sinh a}}.$$

ii) We now use the same trick for the other two integrals:

$$\int_1^\infty \frac{1}{x^2+2x\sinh a-1}\,dx = \int_1^\infty \frac{1}{x^2+(e^a-e^{-a})x-1}\,dx = \int_1^\infty \frac{1}{(x+e^a)(x-e^{-a})}\,dx$$

$$= \frac{1}{(e^a+e^{-a})}\int_1^\infty \frac{1}{x-e^{-a}} - \frac{1}{x+e^a}\,dx$$

$$= \frac{1}{2\cosh a}\Big[\ln(x-e^{-a}) - \ln(x+e^a)\Big]_1^\infty$$

$$= \frac{1}{2\cosh a}\left[\ln\left(\frac{x-e^{-a}}{x+e^a}\right)\right]_1^\infty$$

$$= \frac{1}{2\cosh a}\left[\overset{0}{\cancel{\ln 1}} - \ln\left(\frac{1-e^{-a}}{1+e^a}\right)\right]$$

$$= \frac{1}{2\cosh a}\ln\frac{1+e^a}{1-e^{-a}} = \frac{1}{2\cosh a}\ln\left(\frac{e^{a/2}}{e^{-a/2}}\cdot\frac{e^{a/2}+e^{-a/2}}{e^{a/2}-e^{-a/2}}\right)$$

$$= \frac{1}{2\cosh a}\left(\ln e^a + \ln\left(\frac{\cosh\frac{a}{2}}{\sinh\frac{a}{2}}\right)\right)$$

$$= \frac{1}{2\cosh a}\left(a+\ln\coth\frac{a}{2}\right)$$

$$\int_0^\infty \frac{1}{x^4+2x^2\cosh a+1}\,dx = \int_0^\infty \frac{1}{x^4+(e^a+e^{-a})x^2+1}\,dx = \int_0^\infty \frac{1}{(x^2+e^a)(x^2+e^{-a})}\,dx$$

$$= \frac{1}{e^a-e^{-a}}\int_0^\infty \frac{1}{x^2+e^{-a}} - \frac{1}{x^2+e^a}\,dx$$

$$= \frac{1}{2\sinh a}\int_0^\infty \frac{1}{x^2+e^{-a}}\,dx - \frac{1}{2\sinh a}\int_0^\infty \frac{1}{x^2+e^a}\,dx$$

Let $x = e^{-\frac{a}{2}}\tan\theta$ for the first integral, $x = e^{a/2}\tan\theta$ for the second.

Then $\frac{dx}{d\theta} = e^{-\frac{a}{2}}\sec^2\theta$, $\qquad \frac{dx}{d\theta} = e^{a/2}\sec^2\theta$

$x=\infty \Rightarrow \theta = \frac{\pi}{2}$ $\qquad x=\infty \Rightarrow \theta=\frac{\pi}{2}$

$x=0 \Rightarrow \theta=0$ $\qquad x=0 \Rightarrow \theta=0$

good substitution if you can't quote the integral directly.

So integral is $\frac{1}{2\sinh a}\int_0^{\pi} \frac{1}{e^{-a}(\cancel{\tan^2\theta+1})}\cdot e^{-a/2}\cancel{\sec^2\theta} - \frac{1}{e^a(\cancel{\tan^2\theta+1})}\cdot e^{a/2}\cancel{\sec^2\theta}\,d\theta$

$$= \frac{1}{2\sinh a}\int_0^{\pi} e^{a/2} - e^{-a/2}\,d\theta$$

$$= \frac{1}{2\sinh a}\cdot\pi\cdot\sinh\frac{a}{2}$$

$$= \frac{\pi\cancel{\sinh\frac{a}{2}}}{2\cdot 2\cancel{\sinh\frac{a}{2}}\cosh\frac{a}{2}} = \boxed{\frac{\pi}{4\cosh\frac{a}{2}}}$$