(i) We have \[ C = \frac{1}{n+1} \sum_{k=1}^{n+1} x_k = \frac{1}{n+1} \left[ \frac{1}{n} \sum_{k=1}^{n} x_k + x_{n+1} \right] \]
\[ = \frac{1}{n+1} \left[ nA + x_{n+1} \right]. \]

(ii) We have \[ B = \frac{1}{n} \sum_{k=1}^{n} (x_k - A)^2. \]
\[ = \frac{1}{n} \left[ \frac{1}{n} \sum_{k=1}^{n} x_k^2 - \frac{2}{n} A \sum_{k=1}^{n} x_k + \frac{1}{n} A^2 \right] \]
\[ = \frac{1}{n} \left[ \sum_{k=1}^{n} x_k^2 - 2A \sum_{k=1}^{n} x_k + A^2 \right] \]
\[ = \frac{1}{n} \left[ \sum_{k=1}^{n} (x_k^2) - 2nA^2 + nA^2 \right] \]
\[ = \frac{1}{n} \sum_{k=1}^{n} (x_k^2) - A^2. \]

(iii) We have \[ D = \frac{1}{n+1} \sum_{k=1}^{n+1} (x_k - c)^2 \]
\[ = \frac{1}{n+1} \left[ \sum_{k=1}^{n+1} x_k^2 - 2c \sum_{k=1}^{n+1} x_k + \sum_{k=1}^{n+1} c^2 \right] \]
\[ = \frac{1}{n+1} \left[ \sum_{k=1}^{n+1} (x_k^2) - 2c + c^2 \right] \]
\[ = \frac{1}{n+1} \left[ \frac{1}{n} \sum_{k=1}^{n} (x_k^2) + x_{n+1}^2 \right] - c^2 \]
\[ = \frac{1}{n+1} \left[ n(\bar{x}^2 + A^2) + x_{n+1}^2 \right] - c^2 \]

Now \( c^2 = \frac{n^2A^2 + 2nA x_{n+1} + x_{n+1}^2}{(n+1)^2} \).

So \[ D = \frac{1}{(n+1)^2} \left[ n(n+1)(\bar{x}^2 + A^2) + (n+1)x_{n+1}^2 - n^2A^2 + 2nA x_{n+1} - x_{n+1}^2 \right] \]
\[ = \frac{1}{(n+1)^2} \left[ nA^2 + n(n+1)B + nx_{n+1}^2 - 2nA x_{n+1} \right]. \]
\[
D = \frac{n}{(n+1)^2} \left[ (n+1)B + (A-x_{n+1})^2 \right] \leq A \text{ nice neat form to work with.}
\]

\[
\Rightarrow (n+1)D = \frac{n}{n+1} \left[ (n+1)B + (A-x_{n+1})^2 \right]
\]

\[
= nB + \frac{n}{n+1} (A-x_{n+1})^2. \quad \text{squares are very useful for inequalities.}
\]

\[
\Rightarrow 0 \quad \text{since } \frac{n}{n+1} > 0 \text{ as } n \in \mathbb{N} \text{ and } (A-x_{n+1})^2 \geq 0.
\]

So \((n+1)D \geq nB\), for all values of \(x_{n+1}\).

Consider \(D-B = \frac{n}{n+1}B + \frac{n}{(n+1)^2} (A-x_{n+1})^2 - B\)

\[
= \frac{n}{(n+1)^2} (A-x_{n+1})^2 - B.
\]

\[
\text{Often a good strategy for these kinds of proofs about inequalities.}
\]

Now \(D-B \leq 0\) \(\Leftrightarrow D-B < 0\). Question asks you to prove "if and only if!"

\[
\Leftrightarrow \frac{n}{(n+1)^2} (A-x_{n+1})^2 - B < 0.
\]

\[
\Leftrightarrow (A-x_{n+1})^2 < \frac{B(n+1)}{n}.
\]

\[
\Leftrightarrow A - \sqrt{\frac{B(n+1)}{n}} < x_{n+1} < A + \sqrt{\frac{B(n+1)}{n}}.
\]