$1^2 = \frac{1 \times 2 \times 3}{6}$

$1^2 + 3^2 = \frac{3 \times 4 \times 5}{6}$

$1^2 + 3^2 + 5^2 = \frac{5 \times 6 \times 7}{6}$

Conjecture

$$\sum_{i=1}^{n} (2i-1)^2 = \frac{(2n-1)(2n)(2n+1)}{6}$$

$2i-1$

Proof using induction:

Base case $n=1$

LHS $1^2 = 1$

RHS $\frac{1 \times 2 \times 3}{6} = \frac{6}{6} = 1$

true for $n=1$.

Inductive step

Assume true for $n=k$:

$$\sum_{i=1}^{k} (2i-1)^2 = \frac{(2k-1)(2k)(2k+1)}{6}.$$

When $n = k+1$:

$$\text{LHS: } \sum_{i=1}^{k+1} (2i-1)^2 = \frac{(2k-1)(2k)(2k+1)}{6} + (2(k+1)-1)^2$$

$$= \frac{(2k-1)(2k)(2k+1)}{6} + (2k+1)^2$$

$$= \frac{(2k-1)(2k)(2k+1)}{6} + \frac{6(2k+1)^2}{6}$$

$$= \frac{(2k+1)}{6}\left\{(2k-1)(2k) + 6(2k+1)\right\}$$

$$= \frac{(2k+1)}{6}\left\{4k^2 + 10k + 6\right\}$$

$$= \frac{(2k+1)}{6}\left\{(2k+2)(2k+3)\right\}$$

$$\text{RHS} = \frac{(2(k+1)-1)(2(k+1))(2(k+1)+1)}{6}$$

$$= \frac{(2k+1)(2k+2)(2k+3)}{6}$$

LHS = RHS

$\therefore$ if true for $n=k$, true also for $n=k+1$.

But we showed it was true for $n=1$

$\therefore$ by induction true $\forall n \in \mathbb{N}$.