Stage 3 ★★
Mixed Selection 1 - Solutions

1. How many rectangles?

Suppose there are a horizontal and b vertical lines. The grid of rectangles formed is then $a-1$ rectangles high, and $b-1$ rectangles wide. This means there are $(a-1)(b-1)$rectangles.

If there are a total of 15 lines, the aim is to make $(a-1)(b-1)$ as large as possible with $a+b=15$.

This can be done by considering the different combinations that add to make 15:

a	b	$(a-1)(b-1)$
1	14	$0\times 13=0$
2	13	$1\times 12=12$
3	12	$2\times 11=22$
4	11	$3\times 10=30$
5	10	$4\times 9=36$
6	9	$5\times 8=40$
7	8	$6\times 7=42$

Therefore, the largest number is 42 rectangles, formed by having seven lines in one direction and eight in the other.

Alternatively, you can use completing the square to maximise the quantity:

Since $a+b=15$, $(a-1)(b-1)=(a-1)(14-a)=-a^2+15a-14$.
Then, this is, completing the square, $-\left(a-\frac{15}{2}\right)^2+\frac{169}{4}$.

This is minimised when the square is minimised, which occurs when $a=7$ or $a=8$ (since a must be an integer).
This gives $6\times 7=42$ rectangles.

2. Standing on the table

Suppose Dmitri is d centimetres tall, Clement is c centimetres tall and the table is t centimetres tall. The information in the question tells us that:

$$c + t = d + 80$$
$$d + t = c + 100$$

Then, add these two equations together, which gives:

$$c + d + 2t = c + d + 180$$

Subtracting $c + d$ from both sides gives: $2t = 180$.
Dividing by 2 gives: $t = 90$.

Therefore, the table is 90cm tall.

3. Brothers and sisters

Let b represent the number of brothers in the family and s represent the number of sisters in the family.

Each brother has $b - 1$ brothers, because he is one of the b brothers but is not his own brother, and s sisters. Similarly, each sister has b brothers and $s - 1$ sisters.

The boy has the same number of brothers as sisters, so $b - 1 = s$.
Each sister has half as many sisters as brothers, so the number of sisters she has is half of the number of brothers, so $s - 1 = \frac{1}{2}b$.

Solving by substitution or elimination, $b = 4$ and $s = 3$.
Solving by substitution
We want to find b and s, where $b - 1 = s$ and $s - 1 = \frac{1}{2}b$.

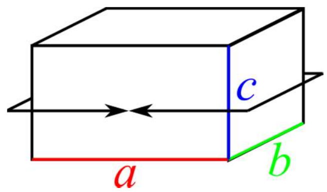
$s - 1 = \frac{1}{2}b$, so $2s - 2 = b$ (by multiplying by 2).

Substituting $b = 2s - 2$ into $b - 1 = s$ gives $2s - 2 - 1 = s$, so $2s - 3 = s$, so $s = 3$.
Substituting $s = 3$ into $b - 1 = s$ gives $b = 4$.
A fuller solution is available at http://nrich.maths.org/12771/solution

These problems are adapted from UKMT (ukmt.org.uk) and SEAMC (seamc.asia) problems.

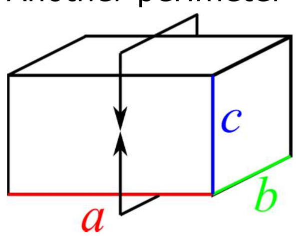
4. Cuboid faces

It will be helpful to label the sides. Here, they are labelled a, b and c.



So this perimeter is equal to $a+b+a+b=2a+2b$.

Another perimeter is shown in the diagram below.



This perimeter is equal to $2a+2c$.

Similarly, the third perimeter will be equal to $2b+2c$.

So we will need to find a, b and c by solving the simultaneous equations

$$2a+2b=12 \Rightarrow a+b=6$$
$$2a+2c=16 \Rightarrow a+c=8$$
$$2b+2c=24 \Rightarrow b+c=12$$

Adding all 3 equations gives $2(a+b+c)=26 \Rightarrow a+b+c=13$.

If $a+b=6$ and $a+b+c=13$, then c must be equal to the difference between 6 and 13 - which is 7.

If $a+c=8$ and $a+b+c=13$, then b must be equal to the difference between 8 and 13 - which is 5.

If $b+c=12$ and $a+b+c=13$, then a must be equal to the difference between 12 and 13 - which is 7.

So the volume is $1 \times 5 \times 7 = 35cm^3$.

A fuller solution is available at: https://nrich.maths.org/12780/solution

5. Square total

Suppose Anastasia thinks of the number a. Then Barry doubles it to get $2a$. Connor triples this to get $6a$ and Damion multiplies it by 6 to get $36a$.

The sum of these is then $a + 2a + 6a + 36a = 45a$. This is a square number.

There are two alternative methods you can use from here.

You could work out the value of $45a$ starting at 1 until you get a square number:

a	$45a$	**Square?**
1	45	No
2	90	No
3	135	No
4	180	No
5	225	Yes: $15^2 = 225$

This shows that the smallest possible number would be 5.

Alternatively, for $45a$ to be a square number, each of its prime factors must be raised to an even power. Since $45 = 3^2 \times 5$ as a product of prime factors, the only prime factor not raised to an even power is 5.

Therefore the smallest value of a that makes this a square must be 5.

These problems are adapted from UKMT (ukmt.org.uk) and SEAMC (seamc.asia) problems.