September 2011: Stage 5: Graphs of Changing Areas

What property do these rectangles share?

The area of every rectangle is 10 square units.

If x is the length of one side, and y is the length of the other side, and $y = {}^{10}/_x$ then $xy = 10$, and therefore the area equals 10.

What symmetry does the graph have? How do you know?

The graph has a line of symmetry in $y = x$. The length and width of the rectangle are interchangeable: when you switch the values of x and y then the new point will be the old point reflected in $y = x$.

What happens to the graph as x gets very large? How do you know?

As the value for x gets very large, the graph will get very close to both the y-axis and the x-axis but will never touch. The y value will become very small as the x axis increases. The lines $y = 0$ and $x = 0$ are asymptotes to the curve of $y = {}^{10}/_x$.

How would the graphs of $y = {}^5/_x$ and $y = {}^{20}/_x$ relate to the one above? Would the graphs intersect? How do you know?

All of the mentioned curves are part of the same "family of graphs". They are all transformations of the standard function: $y = {}^1/_x$.

e.g. $y = {}^5/_x$ $y = {}^{20}/_x$

$y = 5\,({}^1/_x)$ $y = 20\,({}^1/_x)$

These graphs are stretches of $y = {}^1/_x$, with scale factors of 5 and 20 respectively. As the original function of $y = {}^1/_x$ never touches the y-axis or the x-axis, therefore neither will the stretched transformations. Additionally they will never touch each other.

Furthermore each graph has a point on the curve where $y = x$. In this case,

$y = {}^{10}/_x$ $y = {}^{20}/_x$

$y = {}^{10}/_y$ $y = {}^{20}/_y$

$y^2 = 10$ $y^2 = 20$

$y = \sqrt{10}$ $y = \sqrt{20}$

In this way, the graphs are again related. The point at which $y = x$ is where the y and x values are the square root of the transforming stretch factor.

Would you expect the line $y = {}^1/_2\,P - x$ to intersect with the curve $y = {}^{10}/_x$ for all values of P?

Testing by trial and error:

$P=8$,

$y = {}^1/_2\,(8) - x$

$y = 4 - x$

Also, $y = {}^{10}/_{x}$

Therefore,

$4 – x = {}^{10}/_{x}$

$4x - x^2 = 10$

$x^2 - 4x + 10 = 0$

By completing the square,

$(x - 2)^2 - 4 + 10 = 0$

$(x - 2)^2 = -6$,

Therefore, this equation has NO real solutions.

Again, testing by trial and error:

P=20,

$y = {}^{1}/_{2}(20) - x$

$y = 10 - x$

Also, $y = {}^{10}/_{x}$

Therefore,

$10 - x = {}^{10}/_{x}$

$10x - x^2 = 10$

$x^2 - 10x + 10 = 0$

By completing the square,

$(x - 5)^2 - 25 + 10 = 0$

$(x - 2)^2 = 15$,

Therefore, this equation has a solution.

Therefore $8 \leq P \leq 20$

Trial and error continued, until answer found to be between 12 and 13.

Therefore $12 \leq P \leq 13$

Algebraically:

$y = ½P - x$

$y = {}^{10}/_{x}$

Solving simultaneously,

$½P - x = {}^{10}/_{x}$

$\frac{1}{2}Px - x^2 = 10$

$x^2 - \frac{1}{2}Px + 10 = 0$

For solution to the problem, using discriminant, $b^2 - 4ac \geq 0$

Therefore,

$\frac{1}{4}P^2 - 4(1)(10) \geq 0$

$P^2 \geq 160$

$P \geq 12.649$ ….

Therefore, the lines will intersect but only when the value for P is greater than or equal to 12.65 to 2 d.p.

How can you use the graph to find the smallest possible perimeter of a rectangle with an area of 10?

To find the smallest possible perimeter of a rectangle of area 10, you simply draw a straight line of $y = x$. The point at which it crosses the main curve (i.e. $y = {}^{10}/_{x}$) will provide the lengths for the sides of the rectangle with the smallest perimeter.

Testing:

At (√10,√10) the perimeter = 4√10.

At (5,2) the perimeter = 14

At (1,10) the perimeter = 22

Therefore, as the difference between the lengths of the sides of the rectangle increases, two things happen: the area goes down, and the perimeter goes up.

In this example, we are looking at the opposite:

As $y - x \to 0$,

the area → max, and

the perimeter → min

Algebraically:

$y = x^2 - \frac{1}{2}Px + 10$

${}^{dy}/_{dx} = 2x - \frac{1}{2}P$

For smallest perimeter, one needs the minimum point on the line $y = x^2 - \frac{1}{2}Px + 10$, which occurs when ${}^{dy}/_{dx} = 0$.

Therefore,

$2x - \frac{1}{2}P = 0$

$P = 4x$

This suggests that the smallest perimeter occurs where the sides of the rectangle are the same length (i.e $4x$ = 4 lots of the same length side x). This would actually mean that we were considering a square. Again, this would be logical as the difference between the sides is 0 and thus the perimeter is tending towards its minimum.