Irrational Arithmagons

The arithmagon can be written as three algebraic equations:

1) $ab = -6 + 2\sqrt{2}$

2) $ac = 2 - 4\sqrt{2}$

3) $bc = 7 + 7\sqrt{2}$

Where *a*, *b* and *c* represent the sides of the arithmagon

By eliminating the unknowns using substitution, an equation for *a* can be derived:

Using equation 2:

4) $c = (2 - 4\sqrt{2})/a$

By substituting the value of *c* into equation 3:

$$b\left(\frac{2-4\sqrt{2}}{a}\right) = 7 + 7\sqrt{2}$$

5) $b = (a(7 + 7\sqrt{2}))/(2 - 4\sqrt{2})$

Finally, the value of *b* is substituted into equation 1:

$$a\left(\frac{a(7+7\sqrt{2})}{2-4\sqrt{2}}\right) = -6 + 2\sqrt{2}$$

$$\frac{a^2(7+7\sqrt{2})}{2-4\sqrt{2}} = -6 + 2\sqrt{2}$$

Make a^2 the subject and simplify the insides of the brackets

$$a^2 = \frac{(-6+2\sqrt{2})(2-4\sqrt{2})}{7+7\sqrt{2}}$$

$$a^2 = \frac{(-6+2\sqrt{2})(2-4\sqrt{2})}{7+7\sqrt{2}} = \frac{-12+4\sqrt{2}+24\sqrt{2}-16}{7+7\sqrt{2}} = \frac{-28+28\sqrt{2}}{7+7\sqrt{2}} = \frac{-4+4\sqrt{2}}{1+\sqrt{2}}$$

$$a^2 = \frac{-4+4\sqrt{2}}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{-4+4\sqrt{2}+4\sqrt{2}-8}{1-2+\sqrt{2}-\sqrt{2}} = \frac{-12+8\sqrt{2}}{-1} = 12 - 8\sqrt{2}$$

The next step requires some logical guesswork to figure out the square root of $12 - 8\sqrt{2}$

$$a = \sqrt{12-8\sqrt{2}} = 2 - 2\sqrt{2}$$

By substituting the value of *a* into equation 4, *c* can be found:

$$c = \frac{2-4\sqrt{2}}{a} = \frac{2-4\sqrt{2}}{2-2\sqrt{2}} \times \frac{2+2\sqrt{2}}{2+2\sqrt{2}} = \frac{4+4\sqrt{2}-8\sqrt{2}-16}{4-8} = \frac{-12-4\sqrt{2}}{-4} = 3+\sqrt{2}$$

Using equation 1, the value of *b* can be found:

$$ab = -6+2\sqrt{2}$$

$$b = \frac{-6+2\sqrt{2}}{a} = \frac{-6+2\sqrt{2}}{2-2\sqrt{2}} \times \frac{2+2\sqrt{2}}{2+2\sqrt{2}} = \frac{-12-12\sqrt{2}+4\sqrt{2}+8}{4-8-4\sqrt{2}+4\sqrt{2}} = \frac{-4-8\sqrt{2}}{-4} = 1+2\sqrt{2}$$

The values for the vertices of the triangle are:

Top vertex (*b*): $1+2\sqrt{2}$

Lower right vertex (*c*): $3+\sqrt{2}$

Lower left vertex (*a*): $2-2\sqrt{2}$

Check:

$$ab = (2-2\sqrt{2})(1+2\sqrt{2}) = 2-2\sqrt{2}+4\sqrt{2}-8 = -6+2\sqrt{2}$$

$$ac = (2-2\sqrt{2})(3+\sqrt{2}) = 6-6\sqrt{2}+2\sqrt{2}-4 = 2-4\sqrt{2}$$

$$bc = (1+2\sqrt{2})(3+\sqrt{2}) = 3+6\sqrt{2}+\sqrt{2}+4 = 7+7\sqrt{2}$$