1. *INTRODUCTION*

My investigation began, with the search for unsolved problems. Finally, I came across the Millennium Project and found an open-ended problem called "Vecten".

The Problem:[1] *Prove that all four triangles [in the vecten[2]] have the same area.(Part [1]) Construct squares on the outer edges of the triangles and join the outer vertices of the squares to form three quadrilaterals. Find the angles in these quadrilaterals.(Part [2]) Construct another band of squares and quadrilaterals in the same way. Find the outer angles in the quadrilaterals in this band.(Part [3]) If this construction is repeated indefinitely building bands of squares and quadrilaterals, ring upon ring spreading outwards, what else can you discover?(Part [4])*

During my work I aimed at:

- reaching the solution of the original "Vecten" problem
- answering the research question: **What sequence do the lengths of sides of vectens form?**

To achieve the first goal I used concepts including similarity of figures and *Thales' Theorem about ratios of segments*[3]. Further on, to be able to generalise my work, I proved the theorem about parallel sides of a vecten of a degree tending to infinity. I used *proof by induction*. Afterwards, I deduced a formula for the lengths of sides of a vecten of a degree tending to infinity, also with the help of the mathematical induction. Hence, I found the answer for my research question. I also reached the concept of the limit of a sequence to visualise the relation between the initial triangle and the vecten of a degree tending to infinity.

The progress of my analysis revealed that there is a deep relation between lengths of sides of a vecten and terms of *the Fibonacci sequence*[4]. Therefore, I generalised my previous results by proving the explicit formula[5] for the sequences of the form: $E_n = aE_{n-1} + bE_{n-2};\ E_0 = 0;\ E_1 = 1$ for $a^2 + 4\mathrm{b} \geqslant 0;\ n \in \mathbb{N};\ n \geqslant 2$

In the work, introduction gives the text of the problem. The second part explains basic concepts used in the work. The third one contains the solution for the first three points of the original "Vecten" problem. Later, I prove that the sides in infinitely expanded vecten are parallel. The fifth part involves my partial solution for the fourth point, with special attention paid to the sides of vectens. In the last part before conclusion, I obtained the explicit formula for (E_n) .

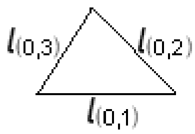
1 University of Cambridge, "Vecten", *Mathematics Enrichment Site*, Rev 09/2005, http://nrich.maths.org/public/viewer.php?obj_id=2862, (12/2007)
2 For definition of vecten see 2.1
3 For theorem see 2.2.2
4 For definition see Appendix 9.1.3
5 For definition see Appendix 9.1.2

2. *BASIC CONCEPTS AND DEFINITIONS*

2.1 Vecten

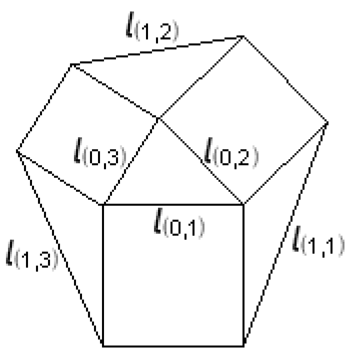
The figure shall be created by construction of squares on the sides of a triangle. The initial triangle is called *the vecten of the 0th degree* (denoted V_0 , Figure 2.1) and the figure created upon the triangle is *the vecten of the 1st degree* (V_1 , Figure 2.2). Vectens of higher degrees are created by constructing squares on the non-square sides of the vecten of the preceding degree (*Vecten of the nth degree* - V_n , Figure 2.3). Notation $l_{(n,k)}; \ k \in [1,2,3]$, where *k* is the index of the side, denotes the segment (the non-square side) located on V_n , whereas $L_{(n,k)}; \ k \in [1,2,3]$ will stand for the length of the non-square side, indexed *k,* of V_n

Figure 2.1 V_0 .



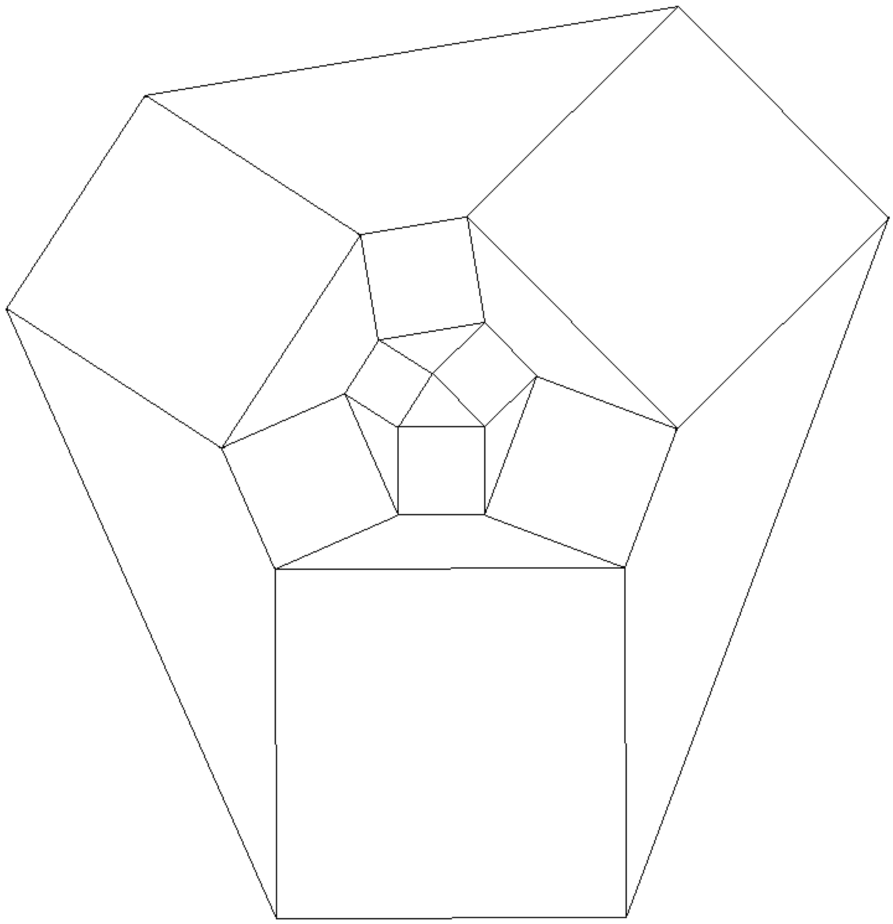
Source: Made by the author.

Figure 2.2 . V_1



Source: Made by the author.

Figure 2.3 . V_3



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2.2 Similarity of figures[6]

2.2.1 Similarity

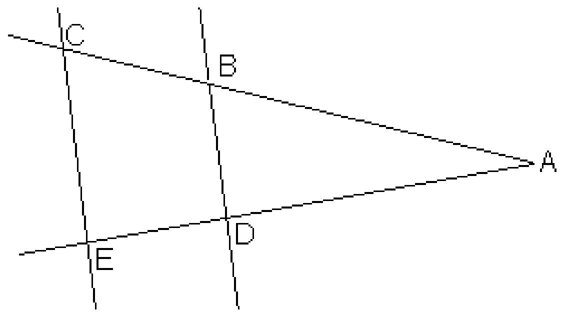
Similarity is a geometrical transformation that leaves all angles intact and changes all distances by the same *scaling factor r*. Similarity relates similar figures. *Translation* (changing of the position) is an example of similarity (r=1) and so is *expansion* (changing size without any relation to the position).

6 Weisstein, Eric W., "Similarity.", *MathWorld--A Wolfram Web Resource,* Rev 8/2002, http://mathworld.wolfram.com/Similarity.html, (12/2007)

2.2.2 Thales' Theorem about the ratios of segments

Theorem[7]**:** If the sides of an angle are crossed by two parallel straight lines, then the segments constructed by the lines on one side of an angle are *similar to* appropriate segments on the other side of an angle. Therefore, (from Figure 2.4) $\frac{AB}{AD}=\frac{BC}{DE}=\frac{AC}{AE}$.

Figure 2.4 . Thales' Theorem



Source: Made by the author.

Thales' Theorem is the basis for the geometrical similarity, since triangles *ABD* and *ACE* are similar to each other by the scaling factor $\frac{AB}{AC}=\frac{AD}{AE}=r$ (From Figure 2.4; all the angles are preserved; from the properties of parallel lines; and all sides are expanded by a factor *r*).

NOTE: In the further part of the work the this theorem will be called Thales' Theorem

2.2.3 Theorem Converse to the Thales' Theorem.

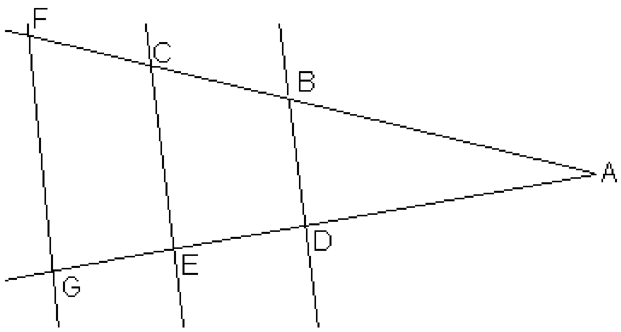
Theorem[8]**:** If $\frac{AB}{AC}=\frac{AD}{AE}$ (Figure 2.4) then lines *BD* and *CE* are parallel.

From this theorem we can derive an important result, used further on in the essay:

7 Interested reader may find the proof (in Polish; unavailable in English) of the Thales' Theorem about the ratios of segments and its Converse on Wikipedia, "Twierdzenie Talesa", Rev 11/2007, http://pl.wikipedia.org/wiki/Twierdzenie_Talesa, (12/2007)

8 See: Footnote 6.

Figure 2.5. Labels for Lemma 1.



Source: Made by the author.

Lemma 1. According to Figure 2.5, if $\frac{BF}{BC}=\frac{DG}{DE}$ and lines BD, CE are parallel, then FG is parallel to both BD and CE.

Proof: From Thales' Theorem applied to lines BD and CE we get $\frac{AB}{AD}=\frac{BC}{DE}=\frac{AC}{AE}$ (1).

We rearrange the equations in the following way.

$\frac{AC}{AE}=\frac{AB}{AD}$, hence, $AC\times AD=AE\times AB$ (2)

Then we rearrange the equation from the thesis:

$\frac{BF}{BC}=\frac{DG}{DE}$, hence, $\frac{BC}{DE}=\frac{BF}{DG}$. Therefore, from (1) we derive: $\frac{BF}{DG}=\frac{AC}{AE}$, hence, $AC\times DG=AE\times BF$. Then we add the resultant equation to (2).

$$AC\times DG=AE\times BF$$
$$AC\times AD=AE\times AB$$

$$AC\times(AD+DG)=AE\times(AB+BF)$$
$$AC\times(AG)=AE\times(AF)$$
$$\frac{AC}{AF}=\frac{AE}{AG}$$

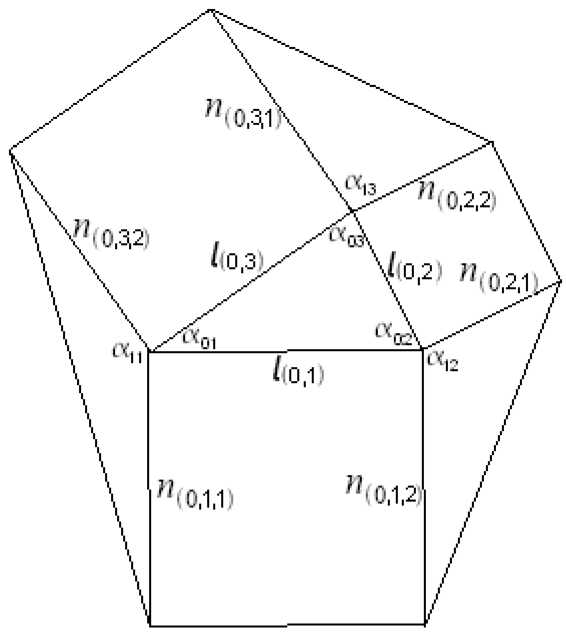
From the Theorem Converse to the Thales' Theorem applied to lines CE and FG we deduce that $BD||CE||FG$ **QED**

3. *SOLUTION OF THE PROBLEM*

In this section I shall solve the problem:
Prove that all four triangles [in the vecten] have the same area (Part [1]). Construct squares on the outer edges of the triangles and join the outer vertices of the squares to form three quadrilaterals. Find the angles in these quadrilaterals (Part [2]). Construct another band of squares and quadrilaterals in the same way. Find the outer angles in the quadrilaterals in this band.(Part [3]), without the last, fourth part, which is open-ended and will be considered later on.

3.1 Solution of Part [1][9]

Figure 3.1 Labels for Part [1].



Source: Made by the author.

Thesis: All four triangles in the vecten (Figure 3.1) have the same area.

Proof: Sides labeled $n_{(0,r,t)}$; $r,t \in \mathbb{N}$ denote segments perpendicular to and of the same length as sides $l_{(0,r)}$. Therefore, directly from the Figure 3.1:

9 Part [1], Part [2] and Part [3] represent parts (sentences) of the problem which directly precede the symbol [1], [2] or [3], respectively.

$360^o = 180^o + \alpha_{03} + \alpha_{13}$
$360^o = 180^o + \alpha_{02} + \alpha_{12}$, hence,
$360^o = 180^o + \alpha_{01} + \alpha_{11}$

$180^o - \alpha_{03} = \alpha_{13}$
$180^o - \alpha_{02} = \alpha_{12}$ (3)
$180^o - \alpha_{01} = \alpha_{11}$

From the formula for the area of a triangle we get:

$$P_1 = \frac{1}{2} L_{(0,3)} L_{(0,1)} \sin\alpha_{01} = \frac{1}{2} L_{(0,1)} L_{(0,2)} \sin\alpha_{02} = \frac{1}{2} L_{(0,2)} L_{(0,3)} \sin\alpha_{03} .$$

According to Figure 3.1 and from the properties of a square:

$|n_{(1,1,1)}| = |n_{(1,1,2)}| = L_{(1,1)}$
$|n_{(1,2,1)}| = |n_{(1,2,2)}| = L_{(1,2)}$ (4)
$|n_{(1,3,1)}| = |n_{(1,3,2)}| = L_{(1,3)}$

Using (4), we derive formulae for areas of other triangles:

$$P_2 = \frac{1}{2} L_{(1,1)} L_{(1,3)} \sin\alpha_{11};\quad P_3 = \frac{1}{2} L_{(1,1)} L_{(1,2)} \sin\alpha_{12};\quad P_4 = \frac{1}{2} L_{(1,2)} L_{(1,3)} \sin\alpha_{13} \quad (5)$$

From (3) and (5) :

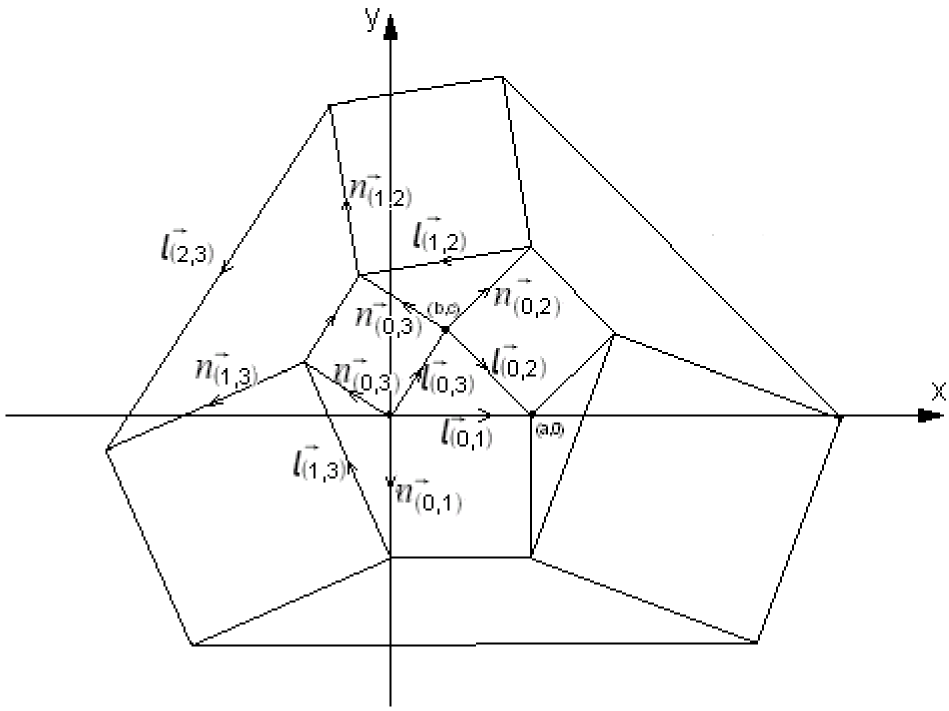
$$P_2 = \frac{1}{2} L_{(0,3)} L_{(0,1)} \sin\alpha_{11} = \frac{1}{2} L_{(0,3)} L_{(0,1)} \sin(180 - \alpha_1) = \frac{1}{2} L_{(0,3)} L_{(0,1)} \sin\alpha_{01} = P_1$$
$$P_3 = \frac{1}{2} L_{(0,1)} L_{(0,2)} \sin\alpha_{12} = \frac{1}{2} L_{(0,1)} L_{(0,2)} \sin(180 - \alpha_2) = \frac{1}{2} L_{(0,1)} L_{(0,2)} \sin\alpha_{02} = P_1$$
$$P_4 = \frac{1}{2} L_{(0,2)} L_{(0,3)} \sin\alpha_{13} = \frac{1}{2} L_{(0,2)} L_{(0,3)} \sin(180 - \alpha_3) = \frac{1}{2} L_{(0,2)} L_{(0,3)} \sin\alpha_{03} = P_1$$

Hence, $P_1 = P_2 = P_3 = P_4$ **QED**

3.2 Solution of Part [2][10]

Firstly, I shall prove Lemma, which is vital for this section.

Figure 3.2 Labels for Lemma 2.



Source: Made by the author.

Lemma 2. $\forall_{k\in\mathbb{N}} \vec{l}_{(2,k)} \parallel \vec{l}_{(0,k)}$.

Proof: I shall only prove it for $k=3$, because for $k=1$ and $k=2$ the proof follows similarly. Let the vertices of the initial triangle have coordinates $(0,0), (a,0), (b,c)$. From the Figure 3.2 we have $\vec{l}_{(0,1)}=\begin{pmatrix} a \\ 0 \end{pmatrix}$; $\vec{l}_{(0,2)}=\begin{pmatrix} a-b \\ -c \end{pmatrix}$; $\vec{l}_{(0,3)}=\begin{pmatrix} b \\ c \end{pmatrix}$.

Therefore, vectors perpendicular to $\vec{l}_{(0,1)}$; $\vec{l}_{(0,2)}$; $\vec{l}_{(0,3)}$ and of the same length have coordinates $\vec{n}_{(0,1)}=\begin{pmatrix} 0 \\ -a \end{pmatrix}$; $\vec{n}_{(0,2)}=\begin{pmatrix} c \\ a-b \end{pmatrix}$; $\vec{n}_{(0,3)}=\begin{pmatrix} -c \\ b \end{pmatrix}$, respectively.

10 Part [1], Part [2] and Part [3] represent parts (sentences) of the problem which directly precede the symbol [1], [2] or [3], respectively.

Thus, the two vectors of sides of V_1 are:

$$\vec{l}_{(1,2)}=\vec{n}_{(0,3)}-\vec{n}_{(0,2)}=\begin{pmatrix}-c\\ b\end{pmatrix}-\begin{pmatrix}c\\ a-b\end{pmatrix}=\begin{pmatrix}-2c\\ 2b-a\end{pmatrix};\ \vec{l}_{(1,3)}=\vec{n}_{(0,3)}-\vec{n}_{(0,1)}=\begin{pmatrix}-c\\ b\end{pmatrix}-\begin{pmatrix}0\\ -a\end{pmatrix}=\begin{pmatrix}-c\\ a+b\end{pmatrix}.$$

Thus, vectors perpendicular to $\vec{l}_{(1,2)};\ \vec{l}_{(1,3)}$ and of the same length correspondingly, have coordinates

$$\vec{n}_{(1,2)}=\begin{pmatrix}2b-a\\ 2c\end{pmatrix};\ \vec{n}_{(1,3)}=\begin{pmatrix}-a-b\\ -c\end{pmatrix}.$$

From the preceding results we can derive:

$$\vec{l}_{(2,3)}=(\vec{n}_{(0,3)}+\vec{n}_{(1,3)})-(\vec{l}_{(0,3)}+\vec{n}_{(0,3)}+\vec{n}_{(1,2)})=\begin{pmatrix}-c\\ b\end{pmatrix}+\begin{pmatrix}-a-b\\ -c\end{pmatrix}-\begin{pmatrix}b\\ c\end{pmatrix}-\begin{pmatrix}-c\\ b\end{pmatrix}-\begin{pmatrix}2b-a\\ 2c\end{pmatrix}=\begin{pmatrix}-4b\\ -4c\end{pmatrix}=-4\begin{pmatrix}b\\ c\end{pmatrix}$$

$\vec{l}_{(2,3)}=-4\vec{l}_{(0,3)}$, hence $\vec{l}_{(2,3)} \parallel \vec{l}_{(0,3)}$

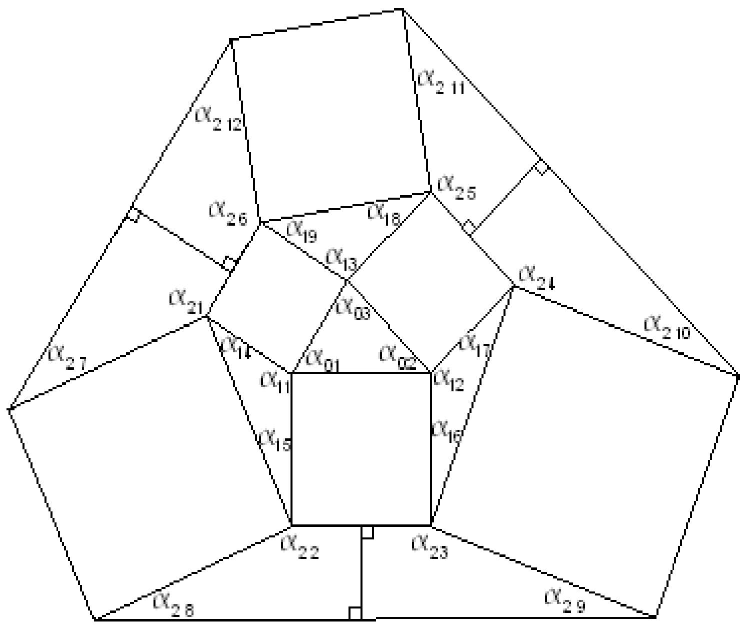
We repeat all steps with the remaining sides of the vecten.

$\forall_{k\in\mathbb{N}} \vec{l}_{(2,k)} \parallel \vec{l}_{(0,k)}$ **QED** (6)

From (6) we can also derive:

Lemma 3. $\forall_{k} L_{(2,k)}=4L_{(0,k)}$.

Figure 3.3 Labels for Part [2], with Lemma 2 included



Source: Made by the author.

From the formula for the sum of angles in a triangle $\begin{matrix}\alpha_{11}+\alpha_{14}+\alpha_{15}=180^o \\ \alpha_{12}+\alpha_{16}+\alpha_{17}=180^o \\ \alpha_{13}+\alpha_{18}+\alpha_{19}=180^o\end{matrix}$.

Substituting $\begin{matrix}180^o-\alpha_{03}=\alpha_{13} \\ 180^o-\alpha_{02}=\alpha_{12} \\ 180^o-\alpha_{01}=\alpha_{11}\end{matrix}$ (3), we obtain

$\begin{matrix}\alpha_{14}+\alpha_{15}=\alpha_{01} \\ \alpha_{16}+\alpha_{17}=\alpha_{02} \\ \alpha_{18}+\alpha_{19}=\alpha_{03}\end{matrix}$, equivalent to

$$\begin{matrix}\alpha_{14}=\alpha_{01}-\alpha_{15} \\ \alpha_{16}=\alpha_{02}-\alpha_{17} \\ \alpha_{18}=\alpha_{03}-\alpha_{19}\end{matrix}$$

Therefore, using Lemma 2,

$$\begin{matrix}\beta=90^o-\alpha_{14}=90^o-(\alpha_{01}-\alpha_{15}) \\ \gamma=90^o-\alpha_{15} \\ \delta=90^o-\alpha_{16}=90^o-(\alpha_{02}-\alpha_{17}) \\ \epsilon=90^o-\alpha_{17} \\ \theta=90^o-\alpha_{18}=90^o-(\alpha_{03}-\alpha_{19}) \\ \phi=90^o-\alpha_{19}\end{matrix} \text{ and}$$

$$\begin{matrix}\alpha_{21}=180^o-\alpha_{14}=180^o-(\alpha_{01}-\alpha_{15}) \\ \alpha_{22}=180^o-\alpha_{15} \\ \alpha_{23}=180^o-\alpha_{16}=180^o-(\alpha_{02}-\alpha_{17}) \\ \alpha_{24}=180^o-\alpha_{17} \\ \alpha_{25}=180^o-\alpha_{18}=180^o-(\alpha_{03}-\alpha_{19}) \\ \alpha_{26}=180^o-\alpha_{19}\end{matrix}.$$

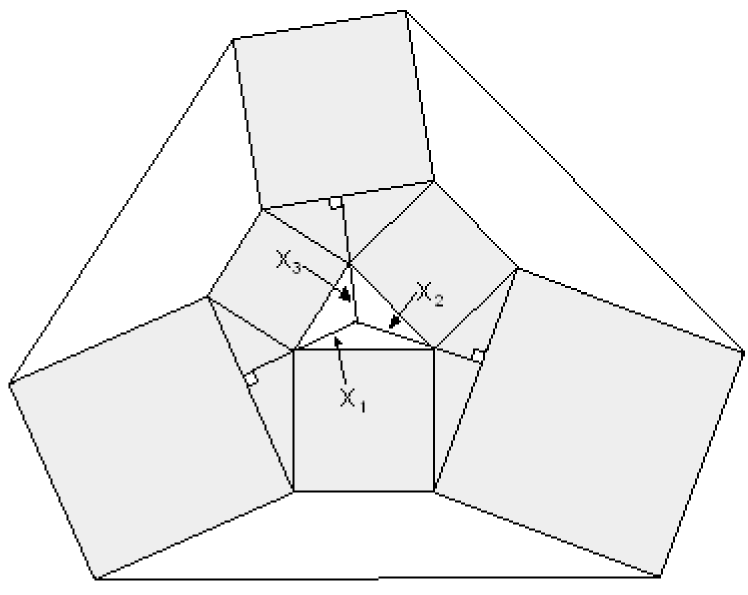
Thus, using the formula for the sum of angles in a triangle,

$$\alpha_{21}=180^o-\alpha_{14}=180^o-(\alpha_{01}-\alpha_{15})$$
$$\alpha_{22}=180^o-\alpha_{15}$$
$$\alpha_{23}=180^o-\alpha_{16}=180^o-(\alpha_{02}-\alpha_{17})$$
$$\alpha_{24}=180^o-\alpha_{17}$$
$$\alpha_{25}=180^o-\alpha_{18}=180^o-(\alpha_{03}-\alpha_{19})$$
$$\alpha_{26}=180^o-\alpha_{19}$$
$$\alpha_{27}=\alpha_{14}=\alpha_{01}-\alpha_{15}$$
$$\alpha_{28}=\alpha_{15}$$
$$\alpha_{29}=\alpha_{16}=\alpha_{02}-\alpha_{17}$$
$$\alpha_{2\,10}=\alpha_{17}$$
$$\alpha_{2\,11}=\alpha_{18}=\alpha_{03}-\alpha_{19}$$
$$\alpha_{2\,12}=\alpha_{19}$$

which is the answer to Part [2].

From Lemma 2. and Lemma 3. we can deduce another important thing, which is presented as Lemma 4.

Figure 3.4 Labels for Lemma 4.

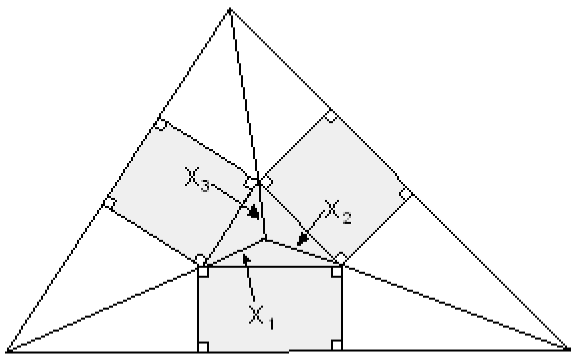


Source: Made by the author.

Lemma 4. $\forall_k L_{(1,k)} = 3x_k$

Proof: By cutting away the shaded parts from Figure 3.4 we obtain the triangle, which is similar to V_0 with the scaling factor $r=4$, according to Lemma 2. and Lemma 3. The result is presented in the Figure 3.5. We observe that after transformation $n_{(1,k)}$ is collinear with x_k (7).

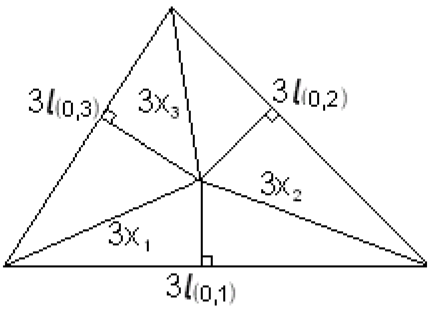
Figure 3.5



Source: Made by the author.

Once again we cut away the shaded rectangles, which are in length equal to $L_{(0,k)}$, and the initial triangle. Thus, we obtain a triangle similar to V_0 with the scaling factor $r=3$ (Figure 3.6) (8)

Figure 3.6

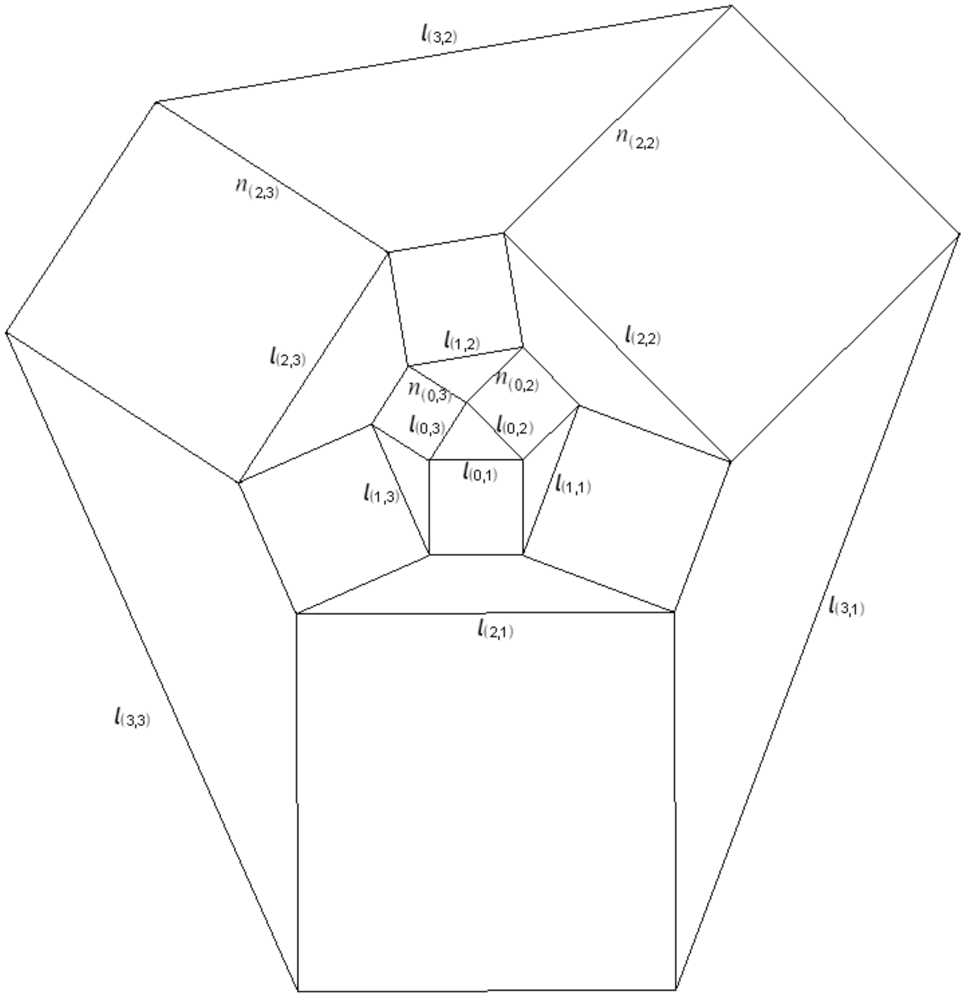


Source: Made by the author.

The triangle we derived is similar to V_0 with the scaling factor $r=3$, basing on the way we constructed x_n. From the last assertion and from results (7) and (8), we deduce that $|n_{(1,k)}| = L_{(1,k)} = 3x_k$ **QED**

3.3 Solution of Part [3][11]

Figure 3.7 Labels for Part [3], with previous results marked.

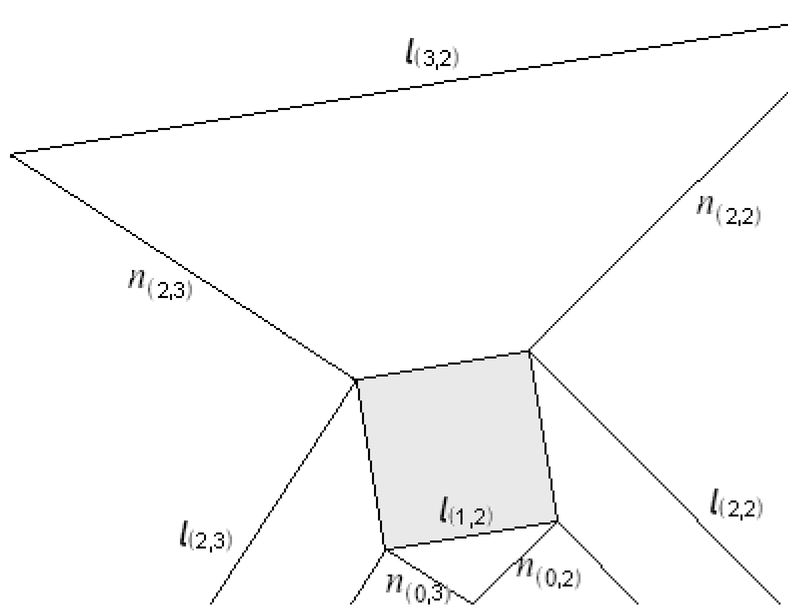


Source: Made by the author.

We will concentrate only on the part of V_3 shown on Figure 3.8, for the solution follows identically for the remaining parts.

11 Part [1], Part [2] and Part [3] represent parts (sentences) of the problem which directly precede the symbol [1], [2] or [3], respectively.

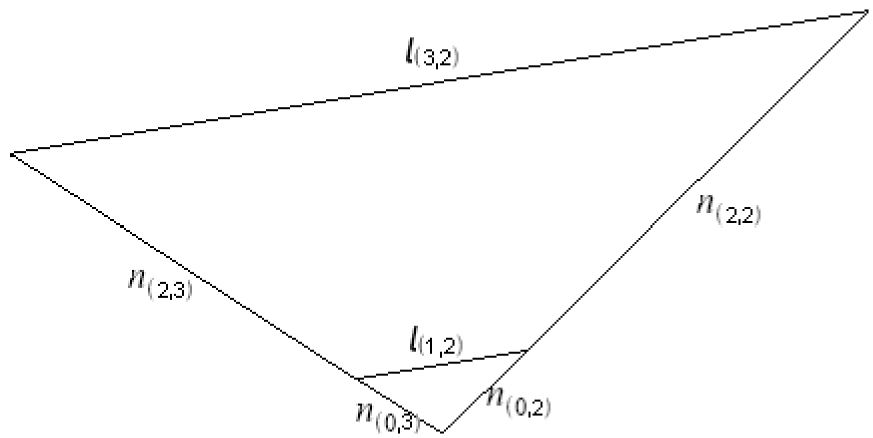
Figure 3.8



Source: Made by the author.

We cut away the shaded square. From Lemma 2 we deduce that after transformation $n_{(0,3)}$ and $n_{(0,2)}$ are collinear with $n_{(2,3)}$ and $n_{(2,2)}$ respectively. Thus, we obtain a triangle (Figure 3.9).

Figure 3.9



Source: Made by the author.

We observe that, according to Lemma 3, $\frac{|n_{(0,2)}|+|n_{(2,2)}|}{|n_{(0,2)}|}=\frac{|n_{(0,3)}|+|n_{(2,3)}|}{|n_{(0,3)}|}=1+\frac{|n_{(2,3)}|}{|n_{(0,3)}|}=5$. Therefore, according to Lemma 1, $l_{(1,2)} \| l_{(3,2)}$, and generally $\forall_k l_{(3,k)} \| l_{(1,k)}$ (9)

Moreover, from Thales' Theorem we can also deduce that:

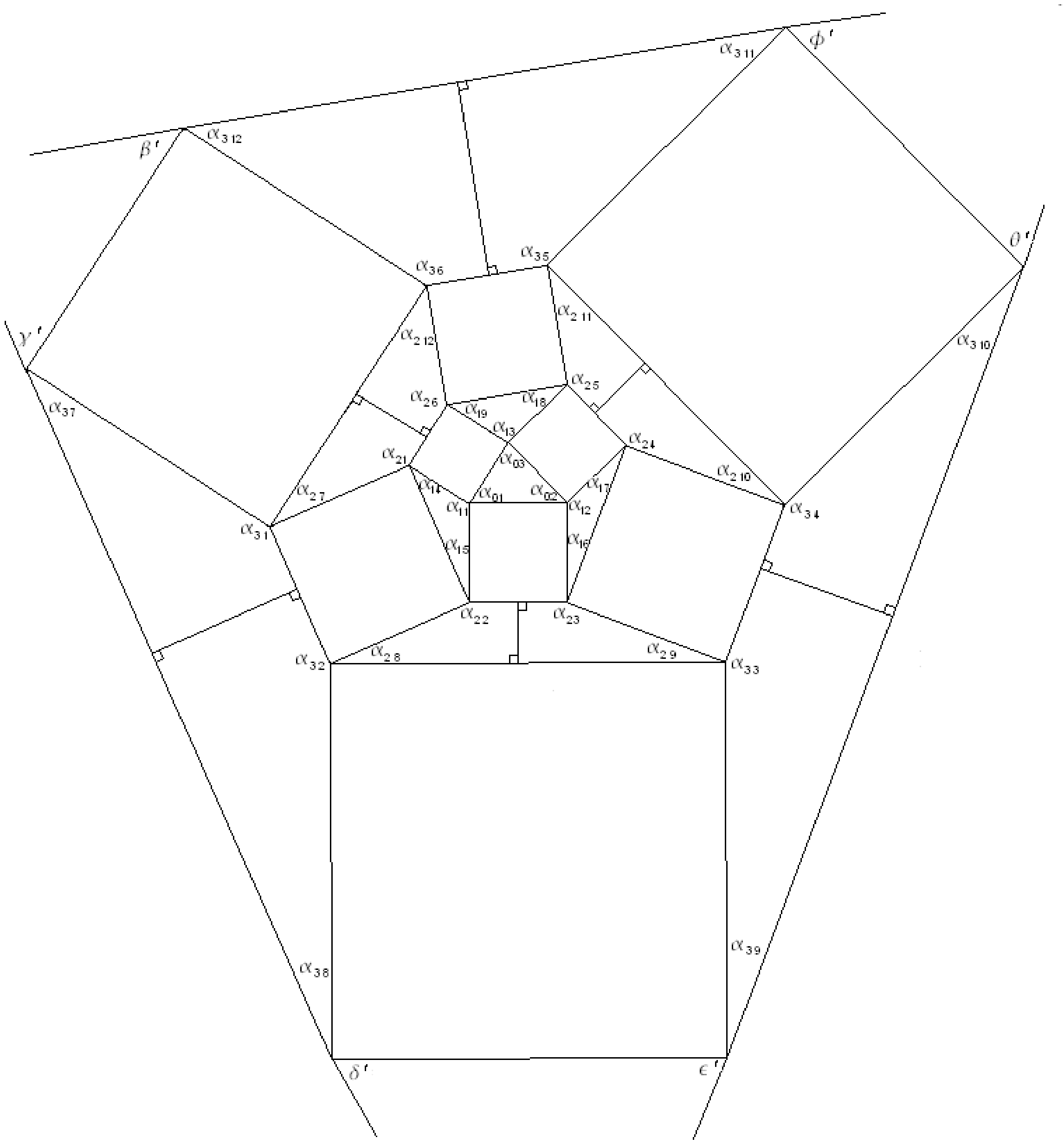
$$\frac{|n_{(0,3)}|}{L_{(1,2)}}=\frac{|n_{(0,3)}|+|n_{(2,3)}|}{|n_{(0,3)}|}\Rightarrow\frac{L_{(3,2)}}{L_{(1,2)}}=1+\frac{|n_{(2,3)}|}{|n_{(0,3)}|}=5 \quad (10)$$

From (10) we deduce

Lemma 5. $\underset{k}{\forall} L_{(3,k)}=5\,L_{(1,k)}$.

Let me remind the appearance of the object of our investigation in the Solution of Part [3].

Figure 3.10 V_3 with (9) included.



Source: Made by the author.

Directly for Figure 3.10 we can deduce -- using the properties of right- angular triangles and the formula for the sum of angles on the straight line -- that the outer angles of V_3 :
$\beta'; \gamma'; \delta'; \epsilon'; \theta'; \phi'$ are equal to:

$$\begin{aligned} &\beta'=90^o-\alpha_{19};\\ &\gamma'=90^o-\alpha_{14};\\ &\delta'=90^o-\alpha_{15};\\ &\epsilon'=90^o-\alpha_{16};\\ &\theta'=90^o-\alpha_{17};\\ &\phi'=90^o-\alpha_{18} \end{aligned},$$

which is the solution for the Part [3] of the problem.

4. *THEOREM ABOUT THE PARALLEL SEGMENTS IN A VECTEN*

In this section I will prove Theorem 1., which is crucial for any further work on the problem.

Theorem 1. $\forall_{n\geqslant 2,k} \; l_{(n,k)} \parallel l_{(n-2,k)}$

Proof of Theorem 1 (by induction):

1^o According to Lemma 2. $\forall_k l_{(2,k)} \parallel l_{(0,k)}$

2^o **Assumption**: $\forall_{n\leqslant m\,,k} \; l_{(n,k)} \parallel l_{(n-2,k)}$

Thesis: $l_{(m+1,k)} \parallel l_{(m-1,k)}$

Proof: Firstly, I shall prove the following Lemma:

Lemma 6: Under the assumption from the inductive proof, $\forall_{n,k}\left(\frac{L_{(2n,k)}}{L_{(0,k)}}=a_{2n} \wedge \frac{L_{(2n+1,k)}}{L_{(1,k)}}=a_{2n+1}\right)$ can

be expressed by the recursive formula $\begin{aligned} a_0=1\;;\; a_1=1\\ a_{2n}=3a_{2n-1}+a_{2n-2};\\ a_{2n+1}=a_{2n}+a_{2n-1};\; n\geqslant 1;\; n\in\mathbb{N}\end{aligned}$.

Proof of Lemma 6 (by induction):

1^o According to Lemma 3 $\underset{k}{\forall} L_{(2,k)}=4L_{(0,k)}$ and $\underset{k}{\forall} L_{(3,k)}=5L_{(1,k)}$. Therefore, we know that

$a_0=\frac{L_{(0,k)}}{L_{(0,k)}}=1 \wedge a_1=\frac{L_{(1,k)}}{L_{(1,k)}}=1 \wedge a_2=\frac{L_{(2,k)}}{L_{(0,k)}}=4 \wedge a_3=\frac{L_{(3,k)}}{L_{(1,k)}}=5$.

This agrees with the values obtained from the formula from the thesis:

$a_0=1 \wedge a_1=1 \wedge a_2=3a_1+a_0=4 \wedge a_3=a_2+a_1=5$.

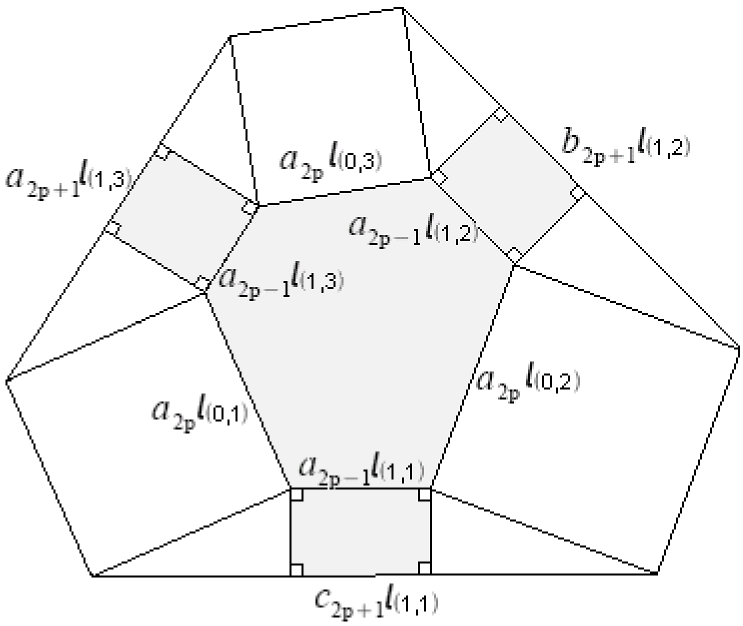
Thus, The formula works for $n=1$

2^o **Assumption**: $a_0=1\ ;\ a_1=1$; $a_{2n}=3a_{2n-1}+a_{2n-2}$; $a_{2n+1}=a_{2n}+a_{2n-1}$; $n\geqslant 1$; $n\in\mathbb{N}$ works for $n\leqslant p\leqslant\frac{m}{2}-1\ ;\ p\in\mathbb{N}$.

Thesis: $a_0=1\ ;\ a_1=1$; $a_{2n}=3a_{2n-1}+a_{2n-2}$; $a_{2n+1}=a_{2n}+a_{2n-1}$; $n\geqslant 1$; $n\in\mathbb{N}$ works for $n\leqslant p+1\leqslant\frac{m}{2}\ ;\ p\in\mathbb{N}$.

Proof: First we construct V_{2p+1} .

Figure 4.1 V_{2p+1}



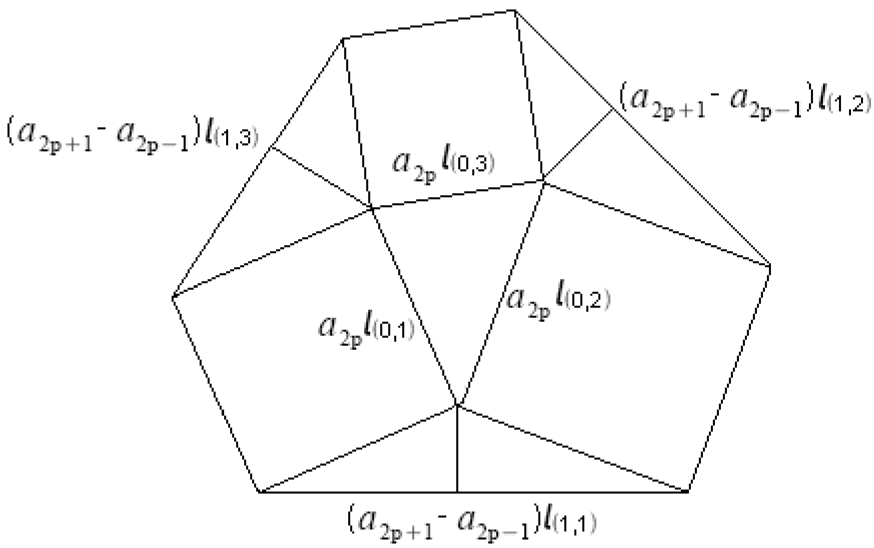
Source: Made by the author.

After cutting away shaded parts on Figure 4.1 we obtain a figure similar to V_1 with a scaling factor $r=a_{2p}$, for we built the vecten on a triangle similar to V_0 with $r=a_{2p}$. Therefore, all scales of sides should be equal (Figure 4.2):

$$a_{2p+1}-a_{2p-1}=b_{2p+1}-a_{2p-1}=c_{2p+1}-a_{2p-1}=a_{2p} \Rightarrow a_{2p+1}=b_{2p+1}=c_{2p+1}=a_{2p}+a_{2p-1}$$

This proves one part of the thesis.

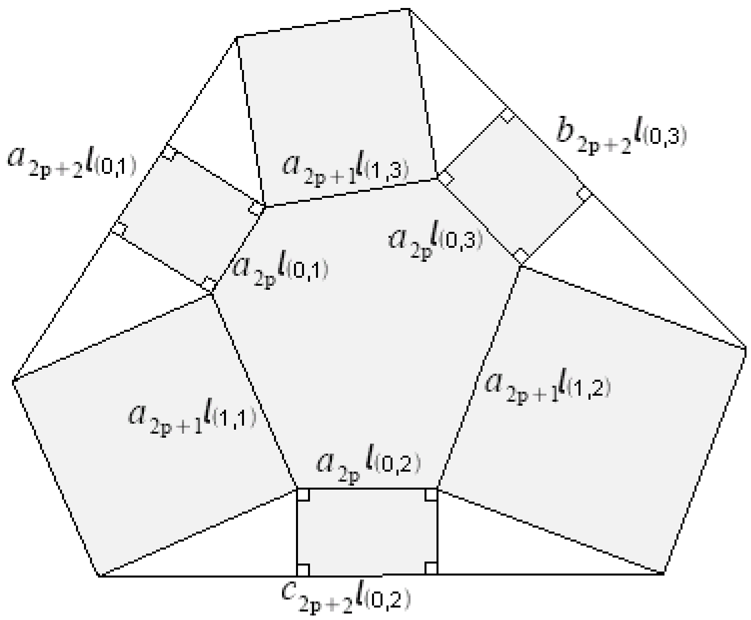
Figure 4.2



Source: Made by the author.

Now we concentrate on the other part of the thesis. Thus, we construct V_{2p+2} (Figure 4.3).

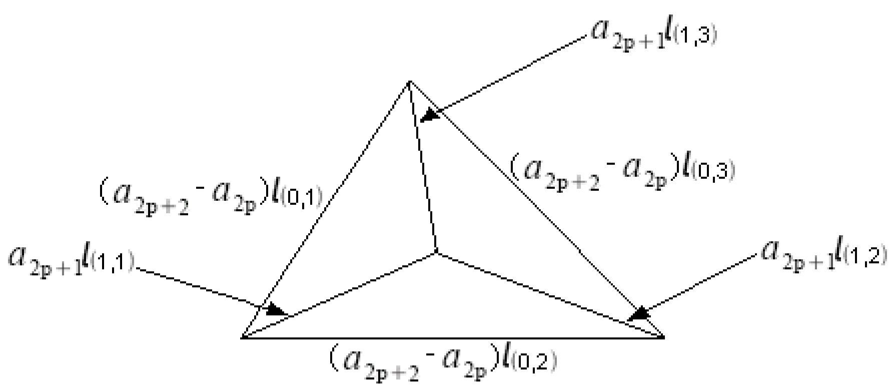
Figure 4.3



Source: Made by the author.

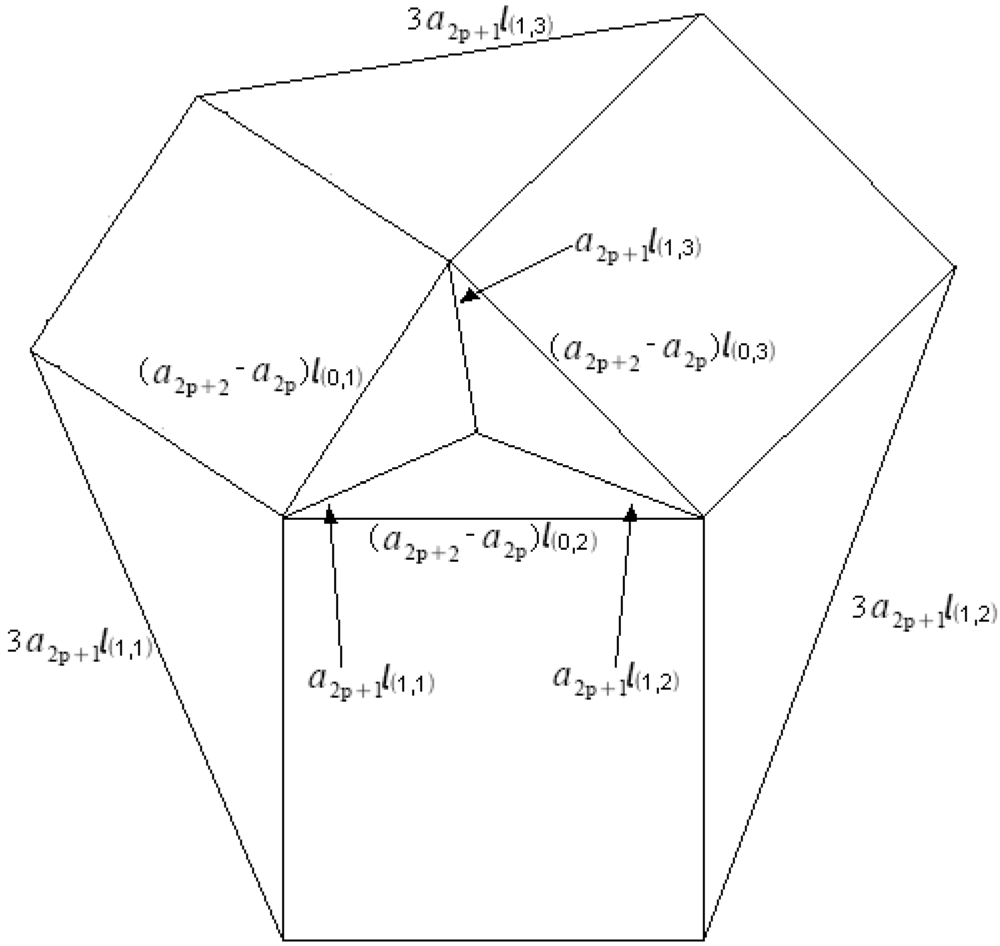
Again we cut away shaded parts, therefore, obtaining a triangle similar to V_0 (thus, it must be true that $a_{2p+2}-a_{2p}=b_{2p+2}-a_{2p}=c_{2p+2}-a_{2p} \Rightarrow a_{2p+2}=b_{2p+2}=c_{2p+2}$, Figure 4.4) where lines inside the object correspond with the segments x_k from Lemma 4 (they are constructed in the same way). Therefore, we construct a vecten on the recently constructed triangle, thus obtaining a vecten similar to V_1 (Figure 4.5).

Figure 4.4



Source: Made by the author.

Figure 4.5



Source: Made by the author.

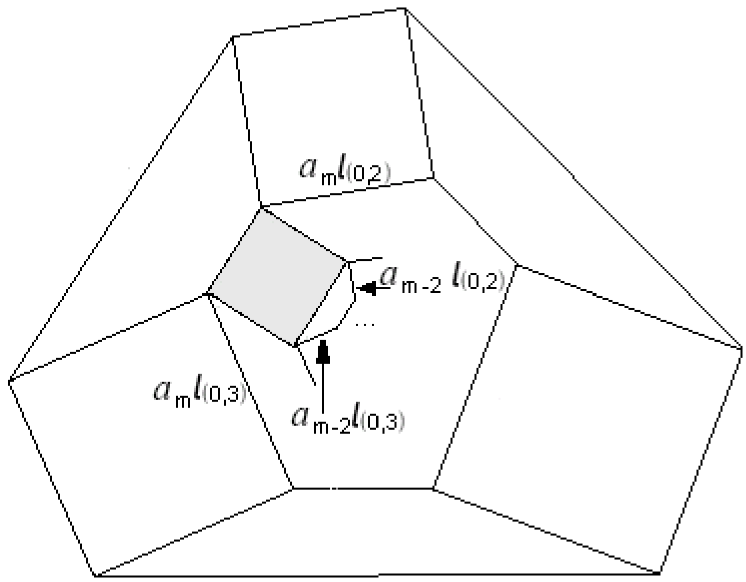
From Lemma 4 we deduce that $y_1 = 3a_{2p+1}l_{1,1}$; $y_2 = 3a_{2p+1}l_{1,2}$; $y_2 = 3a_{2p+1}l_{1,3}$. Therefore, the vecten we got is V_1 scaled by both $r = 3a_{2p+1}$ and $r = 3a_{2p+2} - a_{2p}$, depending at which side we look. Thus we deduce that it must be true that $3a_{2p+1} = a_{2p+2} - a_{2p}$, which is equivalent to $a_{2p+2} = 3a_{2p+1} + a_{2p}$, which proves the second part of the thesis.

We proved that Lemma 6 is valid for $n=1$. Moreover, we derived that on the assumption that Lemma 6 is valid for $n \leqslant p$ it follows that is should be valid for $n \leqslant p+1$; $p \in \mathbb{N}$. Therefore, by the means of the mathematical induction we proved that Lemma 6 works for all $n \in \mathbb{N}$ that satisfy the assumption of the inductive proof of Theorem 1. **QED (Lemma 6)**

Continuation of the Proof of Theorem 1:

Now we construct V_{m+1} (Figure 4.6), using Lemma 6.

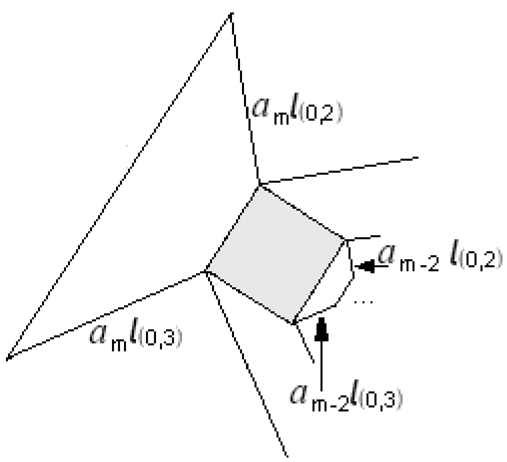
Figure 4.6 V_{m+1}



Source: Made by the author.

We will concentrate only on one part of the vecten (Figure 4.7) to simplify the proof, without loosing its generality.

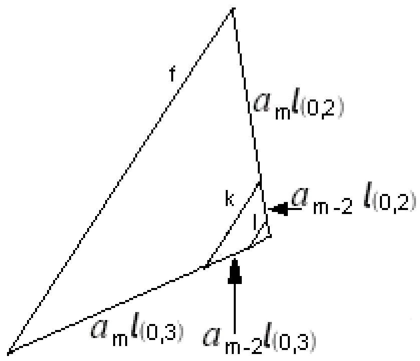
Figure 4.7



Source: Made by the author.

We cut away the shaded part and, thus, get the following object (Figure 4.8):

Figure 4.8



Source: Made by the author.

Basing on the assumption we know that lines l and k are parallel. Moreover, basing on Lemma 6 we can deduce the following equation $\frac{a_{m-2} l_{0,3} + a_m l_{0,3}}{a_{m-2} l_{0,3}} = \frac{a_{m-2} l_{0,2} + a_m l_{0,2}}{a_{m-2} l_{0,2}}$. Thus, from Lemma 1 we deduce that line f is parallel to k . This proves the thesis of the inductive proof of Theorem 1.

We proved that Theorem 1. is valid for $n=2$. Moreover, we derived that on the assumption that Theorem 1 is valid for $n \leqslant m$ it follows that is should be valid for $n \leqslant m+1$ *;* $m \in \mathbb{N}$. Therefore, by the means of the mathematical induction we proved that Theorem 1 works for all $n \in \mathbb{N}$ *;* $n \geqslant 2$. **QED (Theorem 1)**

5. *RELATION BETWEEN* $L_{(n,k)}$ *AND* n *.*

Now I will investigate the sequence mentioned in Lemma 6. and look at the important features of a vecten that result from the existence of the sequence. Firstly, for we proved Theorem 1, we can deduce that Lemma 6. works for all positive integers. Moreover, by the appropriate transformations we can simplify the formula:

$$a_{2n+1}=3a_{2n}+a_{2n-1}$$
$$a_{2n}=a_{2n-1}+a_{2n-2}$$

$$a_{2n+1}=3(a_{2n-1}+a_{2n-2})+a_{2n-1}$$
$$a_{2n+1}=4a_{2n-1}+3a_{2n-2}$$
$$a_{2n+1}=4(3a_{2n-2}+a_{2n-3})+3a_{2n-2}$$
$$a_{2n+1}=15a_{2n-2}+4a_{2n-3}$$
$$a_{2n+1}=15a_{2n-2}+5a_{2n-3}-a_{2n-3}$$
$$a_{2n+1}=5(3a_{2n-2}+a_{2n-3})-a_{2n-3}$$
$$a_{2n+1}=5a_{2n-1}-a_{2n-3} \qquad (11)$$

$$a_{2n}=a_{2n-1}+a_{2n-2}$$
$$a_{2n}=(3a_{2n-2}+a_{2n-3})+a_{2n-2}$$
$$a_{2n}=4a_{2n-2}+a_{2n-3}$$
$$a_{2n}=4(a_{2n-3}+a_{2n-4})+a_{2n-3}$$
$$a_{2n}=5a_{2n-3}+a_{2n-4}$$

$$a_{2n}=a_{2n-3}+4a_{2n-2} \quad /.\times 5$$
$$a_{2n}=5a_{2n-3}+a_{2n-4}$$

$$5a_{2n}=5a_{2n-3}+20a_{2n-2}$$
$$a_{2n}=5a_{2n-3}+a_{2n-4}$$
$$\textit{we substract the equations}$$

$$4a_{2n}=20a_{2n-2}-4a_{2n-4}$$
$$a_{2n}=5a_{2n-2}-a_{2n-4} \qquad (12)$$

We got through these calculations to find recursive formulae, where a term for an odd (even) value of n is defined only by another terms for odd (even) values of *n*. Both (11) and (12) are the same formulae. They only differ in two initial terms defining the sequence. Therefore, we can change Lemma 6. into what is presented as Theorem 2., taking initial values from Lemma 3 and Lemma 5.

Theorem 2. $\forall_{n,k\in\mathbb{Z}^+}\left(\frac{L_{(2n,k)}}{L_{(0,k)}}=b_n \wedge \frac{L_{(2n+1,k)}}{L_{(1,k)}}=c_n\right)$, where $\begin{matrix} b_n=5b_{n-1}-b_{n-2};\ b_0=1;\ b_1=4 \\ c_n=5c_{n-1}-c_{n-2};\ c_0=1;\ c_1=5 \\ n\in\mathbb{N};\ n\geqslant 2\end{matrix}$.

This gives us the important tool for further investigation of the vecten in infinity. However, to fully use this knowledge we should derive an explicit formula, which is much easier to work with. We shall prove that Lemma 7. gives the right formula for (c_n).

Lemma 7. $c_n = \frac{\left(\frac{5+\sqrt{21}}{2}\right)^{n+1} - \left(\frac{5-\sqrt{21}}{2}\right)^{n+1}}{\sqrt{21}}$; $n \in \mathbb{N}$

Proof (by induction):

1^o From explicit formula:

$$c_0 = \frac{\left(\frac{5+\sqrt{21}}{2}\right)^{1} - \left(\frac{5-\sqrt{21}}{2}\right)^{1}}{\sqrt{21}} = \frac{\sqrt{21}}{\sqrt{21}} = 1$$

$$c_1 = \frac{\left(\frac{5+\sqrt{21}}{2}\right)^{2} - \left(\frac{5-\sqrt{21}}{2}\right)^{2}}{\sqrt{21}} = \frac{\frac{20\sqrt{21}}{4}}{\sqrt{21}} = 5$$

$$c_2 = \frac{\left(\frac{5+\sqrt{21}}{2}\right)^{3} - \left(\frac{5-\sqrt{21}}{2}\right)^{3}}{\sqrt{21}} = \frac{\frac{192\sqrt{21}}{8}}{\sqrt{21}} = 24$$

From recursive formula

$$c_0 = 1$$

$$c_1 = 5$$

$$c_2 = 5 \times 5 - 1 = 24$$

All the values agree, therefore, the formula works for $n \leqslant 2$; $n \in \mathbb{N}$

2^o **Assumption**: $c_k = \frac{\left(\frac{5+\sqrt{21}}{2}\right)^{k+1} - \left(\frac{5-\sqrt{21}}{2}\right)^{k+1}}{\sqrt{21}}$; $k \in \mathbb{N}$; $k \geqslant 2$.

Thesis: $c_{k+1} = 5c_k - c_{k-1} = \frac{\left(\frac{5+\sqrt{21}}{2}\right)^{k+2} - \left(\frac{5-\sqrt{21}}{2}\right)^{k+2}}{\sqrt{21}}$; $k \in \mathbb{N}$; $k \geqslant 2$

Proof: For the simplification of the calculations I shall use the following substitution: $A = \frac{5+\sqrt{21}}{2}$; $B = \frac{5-\sqrt{21}}{2}$. I shall prove the thesis by expansion of the LHS using the assumption.

$$c_{k+1}=5c_k-c_{k-1} \underset{Assumption}{=} 5\frac{\left(\frac{5+\sqrt{21}}{2}\right)^{k+1}-\left(\frac{5-\sqrt{21}}{2}\right)^{k+1}}{\sqrt{21}}-\frac{\left(\frac{5+\sqrt{21}}{2}\right)^{k}-\left(\frac{5-\sqrt{21}}{2}\right)^{k}}{\sqrt{21}}$$

$$.=\frac{1}{\sqrt{21}}(5A^{k+1}-A^k-5B^{k+1}+B^k)=\frac{1}{\sqrt{21}}(A^k(5A-1)-B^k(5B-1))$$

$$\frac{1}{\sqrt{21}}(A^k(\frac{25-2+5\sqrt{21}}{2})-B^k(\frac{25-2-5\sqrt{21}}{2}))$$

$$.=\frac{1}{\sqrt{21}}(A^k(\frac{25+21+10\sqrt{21}}{4})-B^k(\frac{25+21-10\sqrt{21}}{4}))=\frac{1}{\sqrt{21}}\left(A^k\left(\frac{5+\sqrt{21}}{2}\right)^2-B^k\left(\frac{5-\sqrt{21}}{2}\right)^2\right)$$

$$.=\frac{A^{k+2}-B^{k+2}}{\sqrt{21}}=\frac{\left(\frac{5+\sqrt{21}}{2}\right)^{k+2}-\left(\frac{5-\sqrt{21}}{2}\right)^{k+2}}{\sqrt{21}}$$

The result is equal to RHS from the thesis.

We proved that Lemma 7. is valid for $n\leqslant 2;\ n\in\mathbb{N}$. Moreover, we derived that on the assumption that Lemma 7 is valid for $n=k;\ k\in\mathbb{N}\ ;\ k\geqslant 2$ it follows that is should be valid for $n=k+1;\ k\in\mathbb{N}\ ;\ k+1\geqslant 3$. Therefore, by the means of the mathematical induction we proved that Lemma 7 works for all $n\in\mathbb{N}$ **QED**

For the work mainly concentrates on the properties of the sides of the vecten, I was interested in the appearance of the vecten in infinity. We need the explicit formula for (b_n) , in order to derive the ratio of the lengths of neighbouring sides. Let's make the following observation. From recursive formulae we can derive:

Figure 5.1 The relation between (b_n) and (c_n)

n	b_n	c_n	c_n-b_n
0	1	1	0
1	4	5	1
2	19	24	5
3	91	115	24
4	436	551	115

Source: Made by the author.

This enables us to draw the following hypothesis $c_n - b_n = c_{n-1} \Rightarrow b_n = c_n - c_{n-1}$. Therefore, we can prove the following formula for (b_n), which can be easily made explicit by substituting explicit formula for (c_n).

Lemma 8. $b_n = c_n - c_{n-1};\ b_0 = 1;\ \ n \in \mathbb{N};\ n \geqslant 1$

Proof (by induction):

1^o From explicit formula:

$$b_0 = 1$$
$$b_1 = c_1 - c_0 = 5 - 1 = 4$$
$$b_2 = c_2 - c_1 = 24 - 5 = 19$$

From recursive formula

$$b_0 = 1$$
$$b_1 = 4$$
$$b_2 = 5 \times 4 - 1 = 19$$

All the values agree, therefore, the formula works for $n \leqslant 2;\ n \in \mathbb{N}$

2^o **Assumption**: $b_k = c_k - c_{k-1};\ \ k \in \mathbb{N}\ ;\ k \geqslant 2$.

Thesis: $b_{k+1} = c_{k+1} - c_k;\ \ k \in \mathbb{N}\ ;\ k+1 \geqslant 3$.

Proof: We can evaluate the LHS using the assumption:

$$b_{k+1} = 5b_k - b_{k-1} \underset{Assumption}{=} 5(c_k - c_{k-1}) - (c_{k-1} - c_{k-2}) = 5c_k - c_{k-1} - 5c_{k-1} + c_{k-2} = [5c_k - c_{k-1}] - [5c_{k-1} - c_{k-2}]$$

Therefore, from Theorem 2. we derive:

$$b_{k+1} = [5c_k - c_{k-1}] - [5c_{k-1} - c_{k-2}] = c_{k+1} - c_k,$$

which is equal to the RHS of the thesis.

We proved that Lemma 8. is valid for $n \leqslant 2;\ n \in \mathbb{N}$. Moreover, we derived that on the assumption that Lemma 8 is valid for $n = k;\ k \in \mathbb{N}\ ;\ k \geqslant 2$ it follows that is should be valid for $n = k+1;\ k \in \mathbb{N}\ ;\ k+1 \geqslant 3$. Therefore, by the means of the mathematical induction we proved that Lemma 8 works for all $n \in \mathbb{N}$ (because b_0 is given as a value). **QED**

Although this is already the answer for my research question, I decided to find the formula for the ratio of the two neighbouring sides. (we assumed that the vecten is of odd degree, however, for even degree, the value will be calculated identically and will be an inverse of the one below; the index $\frac{n-1}{2}$ occurs instead of n because as we remember (b_n) and (c_n) were one sequence, which was divided).

$$\frac{L_{(n-1,k)}}{L_{(n,k)}}=\frac{b_{\frac{n-1}{2}}}{c_{\frac{n-1}{2}}}\frac{L_{(0,k)}}{L_{(1,k)}}=\frac{L_{(0,k)}}{L_{(1,k)}}\left[\frac{c_{\frac{n-1}{2}}-c_{\frac{n-1}{2}-1}}{c_{\frac{n-1}{2}}}\right]=\frac{L_{(0,k)}}{L_{(1,k)}}\left[1-\frac{c_{\frac{n-1}{2}-1}}{c_{\frac{n-1}{2}}}\right]$$

$$.=\frac{L_{(0,k)}}{L_{(1,k)}}\left[1-\frac{\left(\frac{5+\sqrt{21}}{2}\right)^{\frac{n-1}{2}}-\left(\frac{5-\sqrt{21}}{2}\right)^{\frac{n-1}{2}}}{\left(\frac{5+\sqrt{21}}{2}\right)^{\frac{n-1}{2}+1}-\left(\frac{5-\sqrt{21}}{2}\right)^{\frac{n-1}{2}+1}}\right]$$

The limit for $n\rightarrow\infty$ of the formula above can be easily calculated. To simplify the calculations I shall use the following substitution: $A=\frac{5+\sqrt{21}}{2}$; $B=\frac{5-\sqrt{21}}{2}$ and $\frac{n-1}{2}=v$; $\lim_{n\rightarrow\infty} v=\infty$

$$\lim_{n\rightarrow\infty}\left[\frac{b_{\frac{n-1}{2}}}{c_{\frac{n-1}{2}}}\frac{L_{(0,k)}}{L_{(1,k)}}\right]=\lim_{v\rightarrow\infty}\left[\frac{b_v}{c_v}\frac{L_{(0,k)}}{L_{(1,k)}}\right]=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\rightarrow\infty}\left[\frac{b_v}{c_v}\right]=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\rightarrow\infty}\left[1-\frac{A^v-B^v}{A^{v+1}-B^{v+1}}\right]$$

$$.=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\rightarrow\infty}\left[1-\frac{(A-B)\left(A^v+A^{v-1}B+...+AB^{v-1}+B^v\right)}{(A-B)\left(A^{v-1}+A^{v-2}B+...+AB^{v-2}+B^{v-1}\right)}\right]$$

$$.=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\rightarrow\infty}\left[1-\frac{A^v+A^{v-1}B+...+AB^{v-1}+B^v}{A^{v-1}+A^{v-2}B+...+AB^{v-2}+B^{v-1}}\right]$$

Because $A\times B=\frac{(5+\sqrt{21})(5-\sqrt{21})}{4}=1$, all terms will change in the following way $A^uB^w=A^{u-w}A^wB^w=A^{u-w}(AB)^w=A^{u-w}1^w=A^{u-w}$, by analogy $A^uB^w=B^{w-u}$

The first will be used if $u\geqslant w$ and the second if it is otherwise. Therefore,

$$\lim_{v\rightarrow\infty}\left[\frac{b_v}{c_v}\frac{L_{(0,k)}}{L_{(1,k)}}\right]=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\rightarrow\infty}\left[1-\frac{A^v+A^{v-1}B+...+AB^{v-1}+B^v}{A^{v-1}+A^{v-2}B+...+AB^{v-2}+B^{v-1}}\right]$$

$$.=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\rightarrow\infty}\left[1-\frac{A^v+A^{v-2}+...+B^{v-2}+B^v}{A^{v-1}+A^{v-3}+...+B^{v-3}+B^{v-1}}\right]$$

For $B=\frac{5-\sqrt{21}}{2}\leqslant 1$, we deduce that for $v\rightarrow\infty$ the series consisting of consequent powers of B will converge to some limit g.

Thus,

$$\lim_{v\to\infty}\left[\frac{b_v}{c_v}\frac{L_{(0,k)}}{L_{(1,k)}}\right]=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\to\infty}\left[1-\frac{(A^v+A^{v-2}+...+1)+(B^v+B^{v-2}+...+B^2)}{(A^{v-1}+A^{v-3}+...+1)+(B^{v-1}+B^{v-3}+...+B^2)}\right]$$

$$.=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\to\infty}\left[1-\frac{(A^v+A^{v-2}+...+1)+g}{(A^{v-1}+A^{v-3}+...+1)+g}\right]$$

$$.=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\to\infty}\left[1-\frac{(A^v+A^{v-2}+...+1)}{(A^{v-1}+A^{v-3}+...+1)+g}-\frac{g}{(A^{v-1}+A^{v-3}+...+1)+g}\right]$$

For $A=\frac{5+\sqrt{21}}{2}\geqslant 1$, we deduce that for $v\to\infty$ the geometric series consisting of consequent powers A will diverge. Therefore, the term consisting of a constant g divided by the geometric series of A will converge to 0 . Thus, we can omit this term (similarly we will deduce that the geometrical series of $\frac{1}{A}$ converges to 0):

$$\lim_{v\to\infty}\left[\frac{b_v}{c_v}\frac{L_{(0,k)}}{L_{(1,k)}}\right]=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\to\infty}\left[1-\frac{(A^{v-1}+A^{v-3}+...+A)}{(A^v+A^{v-2}+...+1)+Bg}-\frac{g}{(A^v+A^{v-2}+...+1)+Bg}\right]$$

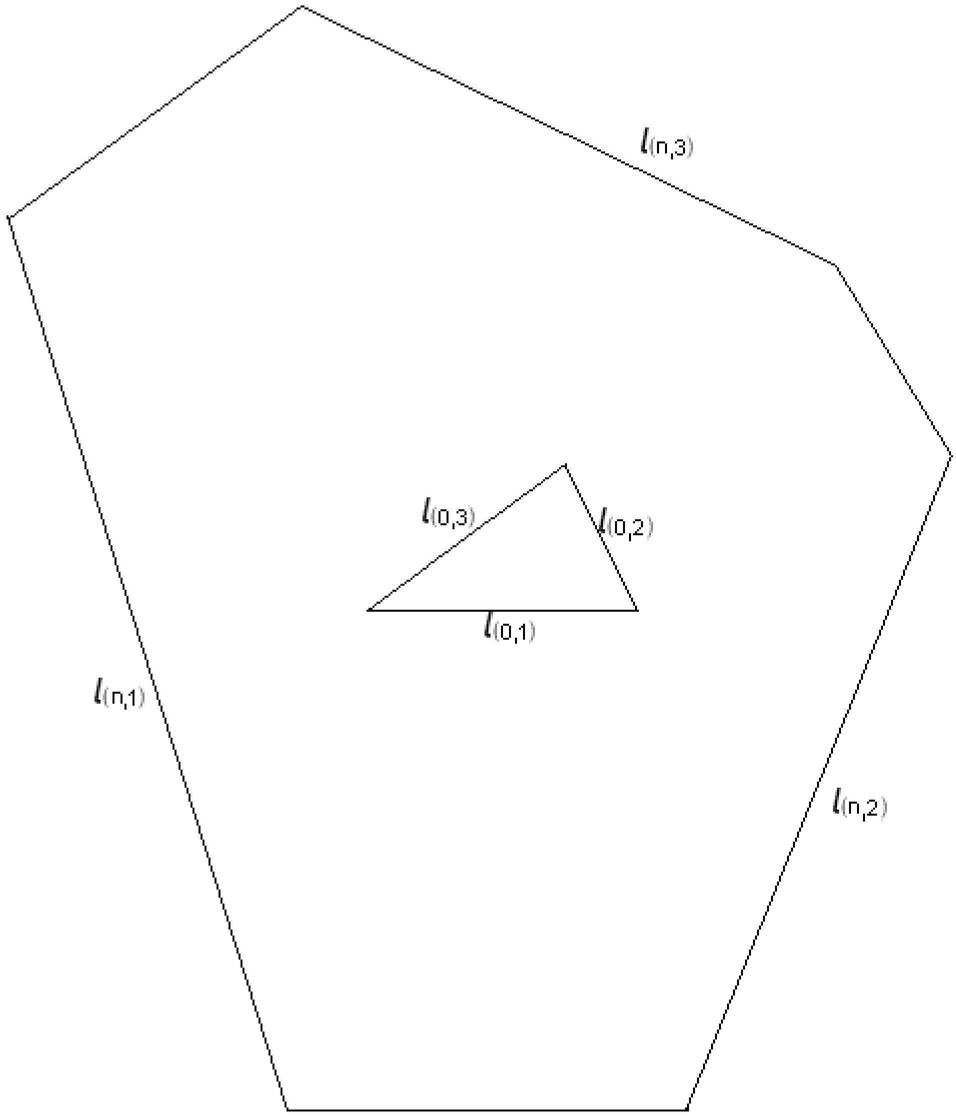
$$.=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\to\infty}\left[1-\frac{(A^{v-1}+A^{v-3}+...+A)}{(A^v+A^{v-2}+...+1)+Bg}\right]=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\to\infty}\left[1-\frac{A^{v-1}}{A^v}\frac{(1+A^{-2}+...+A^{-v})}{(1+A^{-2}+...+A^{-v}+BgA^{-v})}\right]$$

$$.=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\to\infty}\left[1-\frac{A^{v-1}}{A^v}\right]=\frac{L_{(0,k)}}{L_{(1,k)}}\lim_{v\to\infty}\left[1-\frac{1}{A}\right]=\frac{L_{(0,k)}}{L_{(1,k)}}\frac{A-1}{A}=\frac{L_{(0,k)}}{L_{(1,k)}}\frac{\sqrt{21}-3}{2}\approx 0.791\frac{L_{(0,k)}}{L_{(1,k)}}$$

$$\lim_{n\to\infty}\frac{L_{(n-1,k)}}{L_{(n,k)}}=\frac{L_{(0,k)}}{L_{(1,k)}}\frac{\sqrt{21}-3}{2}\approx 0.791\frac{L_{(0,k)}}{L_{(1,k)}}$$

Using this we can draw an example of $\lim_{n\to\infty}V_n$ for a certain V_0 (Figure 5.2). It is useful, because it enables us to observe the properties of an infinitely large object on a limited area. We can also create a vecten fractal by drawing fractals of smaller degrees inside the obtained $\lim_{n\to\infty}V_n$. This links this work to another mathematical area, seemingly unrelated to the initial problem.

Figure 5.2 $\lim_{n \to \infty} V_n$ for a certain V_0. Shows what shape a vecten of a degree tending to infinity should have.



Source: Made by the author.

6. *THE RELATION BETWEEN* (c_n) *AND THE FIBONACCI SEQUENCE*

The striking similarity of the formulae for (c_n) to the f*ormulae for Fibonacci numbers* cannot be left without commentary.

There is a reason for that. They both belong to a group of Recursive Equations of the First Degree and both include only two preceding terms in the formula. Therefore, we can find a general solution for the equations of the type: $E_n = aE_{n-1} + bE_{n-2};\ E_0 = 0;\ E_1 = 1$ for $a^2 + 4\mathrm{b} \geqslant 0;\ n \in \mathbb{N};\ n \geqslant 2$.

This equation is somehow different from both (c_n) and Fibonacci Numbers, for the zeroth terms differ. However, if we put 0 as the zeroth term in both mentioned sequences, they will not change, but will only be shifted by one term. Here is the formula I obtained:

Theorem 3.

$$\left[E_n = a E_{n-1} + b E_{n-2};\ E_0 = 0;\ E_1 = 1;\ a^2 + 4\mathrm{b} \geqslant 0;\ n \in \mathbb{N};\ n \geqslant 2\right] \Leftrightarrow$$

$$\left[E_n = \frac{\left(\frac{a+\sqrt{a^2+4\mathrm{b}}}{2}\right)^n - \left(\frac{a-\sqrt{a^2+4\mathrm{b}}}{2}\right)^n}{\sqrt{a^2+4\mathrm{b}}}\ ;\ a^2+4\mathrm{b} \geqslant 0;\ n \in \mathbb{N}\right]$$

Proof (by induction):

1^o From explicit formula:

$$E_0 = \frac{\left(\frac{a+\sqrt{a^2+4\mathrm{b}}}{2}\right)^0 - \left(\frac{a-\sqrt{a^2+4\mathrm{b}}}{2}\right)^0}{\sqrt{a^2+4\mathrm{b}}} = \frac{0}{\sqrt{a^2+4\mathrm{b}}} = 0$$

$$E_1 = \frac{\left(\frac{a+\sqrt{a^2+4\mathrm{b}}}{2}\right)^1 - \left(\frac{a-\sqrt{a^2+4\mathrm{b}}}{2}\right)^1}{\sqrt{a^2+4\mathrm{b}}} = \frac{\sqrt{a^2+4\mathrm{b}}}{\sqrt{a^2+4\mathrm{b}}} = 1$$

$$E_2 = \frac{\left(\frac{a+\sqrt{a^2+4\mathrm{b}}}{2}\right)^2 - \left(\frac{a-\sqrt{a^2+4\mathrm{b}}}{2}\right)^2}{\sqrt{a^2+4\mathrm{b}}} = \frac{\frac{4a\sqrt{a^2+4\mathrm{b}}}{4}}{\sqrt{a^2+4\mathrm{b}}} = a$$

From recursive formula:

$E_0 = 0$

$E_1 = 1$

$E_2 = a$

All the values agree, therefore, the formula works for $n \in \mathbb{N};\ n \leqslant 2$

2^o **Assumption**: $E_k=\dfrac{\left(\dfrac{a+\sqrt{a^2+4b}}{2}\right)^k-\left(\dfrac{a-\sqrt{a^2+4b}}{2}\right)^k}{\sqrt{a^2+4b}}$; $k\in\mathbb{N}$; $k\geqslant 2$.

Thesis: $E_{k+1}=aE_k+bE_{k-1}=\dfrac{\left(\dfrac{a+\sqrt{a^2+4b}}{2}\right)^{k+1}-\left(\dfrac{a-\sqrt{a^2+4b}}{2}\right)^{k+1}}{\sqrt{a^2+4b}}$; $k\in\mathbb{N}$; $k+1\geqslant 3$.

Proof: Similarly as in the proof of Lemma 6 I shall use the following substitution: $A=\dfrac{a+\sqrt{a^2+4b}}{2}$ and $B=\dfrac{a-\sqrt{a^2+4b}}{2}$ I shall prove the thesis by expansion of the LHS using the assumption.

$$E_{k+1}=aE_k+bE_{k-1}\underset{Assumption}{=}a\frac{\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^k-\left(\frac{a-\sqrt{a^2+4b}}{2}\right)^k}{\sqrt{a^2+4b}}+b\frac{\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^{k-1}-\left(\frac{a-\sqrt{a^2+4b}}{2}\right)^{k-1}}{\sqrt{a^2+4b}}$$

$$.=\frac{1}{\sqrt{a^2+4b}}(aA^k+bA^{k-1}-aB^k-bB^{k-1})=\frac{1}{\sqrt{a^2+4b}}(A^{k-1}(aA+b)-B^{k-1}(aB+b))$$

$$.=\frac{1}{\sqrt{a^2+4b}}\left(A^{k-1}\left(\frac{a^2+a\sqrt{a^2+4b}}{2}+b\right)-B^{k-1}\left(\frac{a^2-a\sqrt{a^2+4b}}{2}+b\right)\right)$$

$$.=\frac{1}{\sqrt{a^2+4b}}\left(A^{k-1}\left(\frac{a^2+2b+a\sqrt{a^2+4b}}{2}\right)-B^{k-1}\left(\frac{a^2+2b-a\sqrt{a^2+4b}}{2}\right)\right)$$

$$.=\frac{1}{\sqrt{a^2+4b}}\left(A^{k-1}\left(\frac{a^2+2\mathrm{a}\sqrt{a^2+4b}+a^2+4b}{4}\right)-B^{k-1}\left(\frac{a^2-2\mathrm{a}\sqrt{a^2+4b}+a^2+4b}{4}\right)\right)$$

$$.=\frac{1}{\sqrt{a^2+4b}}\left(A^{k-1}\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^2-B^{k-1}\left(\frac{a-\sqrt{a^2+4b}}{2}\right)^2\right)$$

$$.=\frac{A^{k+1}-B^{k+1}}{\sqrt{a^2+4b}}=\frac{\left(\frac{a+\sqrt{a^2+4b}}{2}\right)^{k+1}-\left(\frac{a-\sqrt{a^2+4b}}{2}\right)^{k+1}}{\sqrt{a^2+4b}}$$

The result is equal to RHS from the thesis.

We proved that Theorem 3. is valid for $n \in \mathbb{N};\ n \leqslant 2$. Moreover, we derived that on the assumption that Theorem 3 is valid for $n=k;\ k \in \mathbb{N}\ ;\ k \geqslant 2$ it follows that is should be valid for $n=k+1;\ k \in \mathbb{N}\ ;\ k+1 \geqslant 3$. Therefore, by the means of the mathematical induction we proved that Theorem 3 works for all $n \in \mathbb{N}$ **QED**

This is a very important generalisation of the problem, for some types of recursive equations are still unsolved today. It also shows how interconnected mathematics is, how seemingly distant areas like geometry and analysis are closely connected.

7. *CONCLUSION*

The primary aim of my extended essay was to determine the formula of the sequence formed by the lenghts of sides of V_n with respect to V_0. This goal has been fully achieved. The extended essay is based on the problem take for the Millennium Project website.

Throughout the work I managed to:

- solve the problem stated on the website
- derive Theorem 1 stating that respective sides in consequent vectens are parallel, which is an indispensable tool for any analysis of an infinite vecten
- find recursive and explicit formulae for the sequence formed by the lenghts of sides of V_n and, therefore, answer my research question: **What sequence do the lengths of sides of vectens form?**
- observe the similarity of derived formulae to the formulae for the Fibonacci sequence and
- thus, by analogy, derive the general formula for sequences like (E_n).

As my analysis progressed I observed some other interesting properties of V_n, which unfortunately extend beyond the boundaries of this essay. For instance, I observed that the quadrilaterals formed by two consequent vectens have the same areas, which, additionally form a sequence $1,\ 5,\ 24,\ 115,\ldots$, which is exactly the same as (c_n).

My results also forced me to draw a following hypothesis: **If we create a "vecten" on a quadrilateral or an n-sided polygon, the sides of such a figure shall form a sequence like** (E_n)**, but with different coefficients then** (c_n), which could be my new research question. However, this extends well beyond the limit of this essay and, thus I dropped this topic. Nevertheless, it remains unsolved problem for the further stage of my education.

In my opinion, I fulfilled the requirements of this essay by answering the research question and afterwards obtaining the important generalisation. Furthermore, the problem I investigated seems to have many more unsolved generalisations, which, however, cannot be fitted into the limit of the extended essay.

8. *SOURCES*

[1] University of Cambridge, “Vecten”, *Mathematics Enrichment Site*, Rev 09/2005, http://nrich.maths.org/public/viewer.php?obj_id=2862, (12/2007)

[2] Weisstein, Eric W., "Similarity.", *MathWorld--A Wolfram Web Resource, Rev 8/2002*, http://mathworld.wolfram.com/Similarity.html, (12/2007)

[3] Weisstein, Eric W., "Recursion.", *MathWorld--A Wolfram Web Resource, Rev 8/2002*, http://mathworld.wolfram.com/Recursion.html, (12/2007)

[4] Wikipedia, "Twierdzenie Talesa", Rev 11/2007, http://pl.wikipedia.org/wiki/Twierdzenie_Talesa, (12/2007)

9. *APPENDIX*

9.1 Definitions- Sequences

9.1.1 Recursive formula for a sequence[12]

It is a formula defining a sequence by defining n^{th} term of the sequence only by its relation with the previous terms of the sequence e.g. $a_n = a_{n-1} + a_{n-2};\ a_0 = 1;\ a_1 = 1;\ n \in \mathbb{N};\ n \geqslant 2$. As the example shows, we always have to define the first few terms explicitly, so that we can derive the succeeding terms basing on these values.

9.1.2 Explicit formula for a sequence

It is a formula defining a sequence by defining n^{th} term of the sequence only by its relation to n e.g. $b_n = 2^n;\ n \in \mathbb{N}$. It is very useful to derive an explicit formula of a sequence, because it simplifies the further analysis of the sequence. For instance, it often simplifies finding the limit of the sequence.

9.1.3 Fibonacci sequence

It is a sequence, where a term is created by adding two preceding terms. The 0^{th} and the 1^{st} terms are equal to 1. Recursive formula was given by Fibonacci:

$$F_n = F_{n-1} + F_{n-2};\ F_0 = 1;\ F_1 = 1;\ n \in \mathbb{N};\ n \geqslant 2,$$

whereas explicit formula was given by Binet

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}};\ n \in \mathbb{N}.$$

12 Weisstein, Eric W., "Recursion.", *MathWorld--A Wolfram Web Resource, Rev 8/2002*, http://mathworld.wolfram.com/Recursion.html, (12/2007)

9.2 List of Theorems and Lemmas derived in the work.

9.3 List of Minor Results

9.4 List of Figures

9.5 List of Footnotes

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1	University of Cambridge, “Vecten”, *Mathematics Enrichment Site*, Rev 09/2005, http://nrich.maths.org/public/viewer.php?obj_id=2862, (12/2007)	1
2	For definition of vecten see 2.1	1
3	For theorem see 2.2.2	1
4	For definition see Appendix 9.1.3	1
5	For definition see Appendix 9.1.2	1
6	Weisstein, Eric W., "Similarity.", *MathWorld--A Wolfram Web Resource, Rev 8/2002*, http://mathworld.wolfram.com/Similarity.html, (12/2007)	3
7	Interested reader may find the proof (in Polish; unavailable in English) of the Thales' Theorem about the ratios of segments and its Converse on Wikipedia, "Twierdzenie Talesa", Rev 11/2007, http://pl.wikipedia.org/wiki/Twierdzenie_Talesa, (12/2007)	4
8	See: Footnote 6.	4
9	Part [1], Part [2] and Part [3] represent parts (sentences) of the problem which directly precede the symbol [1], [2] or [3], respectively.	6
10	Part [1], Part [2] and Part [3] represent parts (sentences) of the problem which directly precede the symbol [1], [2] or [3], respectively.	8
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12	Weisstein, Eric W., "Recursion.", *MathWorld--A Wolfram Web Resource, Rev 8/2002*, http://mathworld.wolfram.com/Recursion.html, (12/2007)	35