Mathematics enrichment: what is it and who is it for?

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NRICH Mathematics Project (www.nrich.maths.org)

1. Abstract

The NRICH Project (www.nrich.maths.org) has been in operation since 1996. Its original purpose was to support able young mathematicians whose access to mathematics related opportunities in their local community was limited. The intention was to offer an online community where they could talk about and do mathematics with like-minded youngsters and this remains a principle aim of the project today. Since 1996 the resources on the web site have grown and the project has developed a reputation for creative thinking in the area of mathematics enrichment both nationally and internationally.

Over the last two years we have been carrying out a review of the project with the purpose of identifying ways of using and developing existing resources and creating new ones that are in keeping with our views on enrichment. This paper reflects the findings of this study, which is still underway and which has been running in parallel with the review and development cycle. I have used data from discussions and interviews and artefacts (problems and resources) alongside a literature review support the view of enrichment curriculum offered in this article. The research and development process has resulted in a questioning of our initial understandings of mathematics enrichment as well as views expressed within the current literature. The study identified the need to give coherence to terms such as "problem solving" and "mathematical thinking" as concepts which were felt to underpin "enrichment". I present here an overview of these aspects of the study, alongside our views of teaching for enrichment and learning through enrichment and whether these relate solely to the most able.

2. Introduction

The literature on enrichment, problem solving and mathematical thinking lacks clarity because it fails to give consistent interpretations of each of the terms. Often the terms are used in a way which assumes the reader's interpretation of the phrases "problem solving" or "mathematical thinking" are the same as the author's, although Schoenfeld (1992) is a rare case where a distinction is made, although not a distinction shared by this paper. The word "enrichment" is almost exclusively used in the context of provision for the mathematically most able, with a few notable exceptions such as Wallace (1986). The implication taken from this general lack of clarity, is that the concepts are tightly bound and therefore difficult to separate. My aim is to place them into a meaningful framework to inform future discussion. There are three questions which I will take from this:

- What, if any, is the difference between the terms "enrichment", "mathematical thinking" and "mathematical problem solving"?
- If there is distinction between these terms, can we place them in some meaningful relationship with each other?
- Are the processes they represent simply the domain of the mathematically most able?

3. Defining a Framework

The experience of most students "doing mathematics" involves studying materials and working through tasks set by others (Olkin and Schoenfeld 1994) with very little room for the entrepreneur or creative thinker.

One of the things we at NRICH are trying to offer in an enrichment curriculum is the opportunity to experience:

"The joy of confronting a novel situation and trying to make sense of it - the joy of banging your head against a mathematical wall, and then discovering that there may be ways of either going around or over that wall"

Page 43 Schoenfeld (1994)

However, to achieve this, we need to know what will make this happen. It is therefore necessary to make explicit the content and nature of the skills, knowledge and classroom experience and identify the methodologies for implementation. What do the curriculum resources and learning tools look like that help students come to "know" in this way?

Enrichment is associated by some with models of curriculum acceleration or compaction. However, I do not describe either of these curriculum approaches as enrichment. Acceleration means that students are given access to "standard curriculum material" earlier and are encouraged to move more quickly through subject content, normally resulting in early entry to nationally recognised examinations. Curriculum compaction (Renzulli and Reis 1999) more usually restructures a curriculum in a way that enables students to cover aspects of the standard curriculum "more efficiently". This is often accomplished by clustering related curriculum themes vertically. In the cases of acceleration and compaction, the result is the release of time which can then be used for extra curricular activities. It is not the acceleration or the compaction that concerns enrichment but what happens in the time created by such approaches and "doing more of the same" is not enrichment. Sometimes the time created is used for activities involving problem solving and/or curriculum extension (where students work on mathematics that they will generally not meet in the standard curriculum). However, here the model of enrichment is still viewed as an add on. This denies the vast majority of students access to enrichment opportunities and imposes significant organisational pressures within the classroom. There is also little evidence available to suggest that schemes of acceleration result in any long term benefit to the students involved. If problem solving and mathematical thinking linked to stimulating mathematical contexts is worth doing, surely it is worth doing for everyone a significant proportion of the time. Enrichment should pervade the curriculum as a whole and not simply be available to those who work fastest. If this is to be the case we need not only to consider enrichment as a vehicle for the most able but, for the sake of classroom management, if it can offer something to the vast majority of students it will, at the very least make it a more realistic option. This does have very significant issues for the type and scope of resources that are used to support enrichment activities and the development of materials forms part of the research upon which this paper is based. Here though I will focus first on what we mean by problem solving and mathematical thinking.

3.1 Problem Solving and Mathematical Thinking?

The aim of this section of the paper is to offer a definition of enrichment. I will first spend some time reviewing what I am taking as the constituent concepts of an enrichment curriculum including problems, problem solving. Then I will briefly consider some implications for teaching and learning and finally I will present a framework for enrichment.

3.1.1 *The problem and the problem solver*

The view of problems offered by some authors as "authentic" or "real world" (Blum and Niss 1991), or "word" problems (Riley, Greeno et al. 1983; Doyle 1988; Mayer 2002) is not seen as core to the nature of a problem, but simply contexts in which problems are offered. That is not to forget that it is crucial for problems to engage and "inspire" the student to think the struggle of problem solving is worthwhile. Neither is this simply about the problems in isolation but also about pupils and teaching approaches that encourage a non-mechanical or simplistic view of mathematics.

Many authors (Polya 1957; Riley, Greeno et al. 1983; Mayer 2002) describe a problem as a problem because it has a goal (the solution) and the student does not have the "right" sort of knowledge to be able to solve it immediately. The "right" knowledge is both the mathematical concept knowledge and the problem solving knowledge. Schoenfeld, and Blum and Niss, Schoenfeld (1985; 1991; 1992) refer to problems as being "relative" to the individuals knowledge and experience.

"A problem is not inherent in a task but depends on the individual."

Page 74 Schoenfeld (Schoenfeld 1985)

Referring to the report of the Problem Solving Theme Group at the ICME 5, Mason and Davis (1991) stress the importance of the autonomy of the solver in terms of what they try to do and what constitutes for them a satisfactory end point. This centring on the solver is reflected in the paper by Pape, Bell et al (2003) who suggest that pupils should see themselves as agents in their own learning when problem solving.

It is this idea of a problem as something that needs to be worked on without a clear idea on the part of the solver, at least to start with, about how to find the solution that is key. However, for a problem to be effective it also needs to engage, be accessible (enable the pupil to step into the problem) and be accompanied by the necessary support to help progress towards a solution with the pupil leading rather than being led. No matter how good on a superficial level a problem might be (relevant, timely, well structured, clear) if the student cannot engage with it because it starts at a point beyond their zone of proximal development (Vygotsky 1978) then no amount of support will result in a learning gain. If, in addition it does not offer scope for the varying learning needs and levels of knowledge of a group of students then it is unlikely to be of value in a classroom setting. In this case some students will not be able to engage at all some will eventually solve the problem, and the most able will have finished five minutes after the start of the lesson, because to them the problem was not even a problem.

3.1.2 *Problem solving*

"more important than specific mathematical results are the habits of mind used by people who create those results"

Cuoco, Goldenberg et al. Page 375 (1996)

There have been a number of attempts to describe the problem solving process, possibly starting in 1933 by Dewey with his five stage model. These five stages have much in common with the work of Polya (1957), who expounded and related a four element model. Those four elements were; understanding the problem, devising a plan, carrying out the plan and looking back. Polya also went on to look at particular techniques that can prove useful within each of these elements. More recently Mayer (2002), who offers a similar model to Polya's in his "componential theory of problem solving, considers similar "elements". Much of the problem solving literature either develops of reiterates these basic "elements" and guidance in tackling mathematical problems.

In essence I am advocating problem solving as an iterative process based on identifiable elements, similar to those of Polya and others including Wilson, Fernandez et al. (C1991 ??), Mason et al (1985), Mayer (2002), Ernest (2000). That is not to suggest that many of these models were intended to be purely linear, as Wilson et al. (ibid.) suggest with reference to Polya's four element model. Hence the deliberate choice of the term "element" rather than "stage" in this text. These references have many common threads and have models of the process broken down into a varied number of elements. The elements outlined below combine a number of the features of these existing models with our own research findings. This, it must be emphasised, is meant to convey a sense of direction but is not linear. The intention is that the solver may need to revisit elements and reformat the problem as they work towards a solution– giving an iterative feel to the problem solving process.

The C.A.P.E. model

Comprehension

- Making sense of the problem/retelling/creating a mental image,
- Applying a model to the problem;

Analysis and synthesis

- Identifying and accessing required pre-requisite knowledge,
- Applying facts and skills, including those listed in mathematical thinking (above),
- Conjecturing and hypothesising (what if);

Planning and execution

- Considering novel approaches and/or solutions

- Identifying possible mathematical knowledge and skills gaps that may need addressing,
- Planning the solution/mental or diagrammatic model,
- Execute;

Evaluation

- Reflection and review of the solution,
- Self assessment about ones own learning and mathematical tools employed,
- Communicating results.

3.1.3 *The roles of Problem solving*

The place of problem solving is a key one not only in terms of the skills it is possible to develop but also in its role as supporting learning in other areas. These types of generic teaching and learning contexts for problem solving include considering it as a skill to be learned or as a tool to learn through. Authors who consider these roles for problem solving in more depth include: Stanic and Kilpatrick (1988), Nunokawa (2004); Wilson, Fernandez et al. (1991) Blum and Niss (1991) . From this literature it is possible to identify the following roles:

1. Problem solving as a means of learning mathematical content – teaching through problem solving.
2. Problem solving in order to learn about the processes of problem solving (the formative argument of Blum and Niss) – teaching about.
3. Problem solving as a generic skill applicable to other subjects as well as mathematics and offering the ability to take a critical view of the world (utility argument) - teaching for.
4. Problem solving as a motivational tool – to give relevance to other aspects of mathematics.
5. Problem solving as a fundamental part of mathematics.

Although this list merits a much more detailed discussion its inclusion here is simply to highlight the range of reasons we might engage in problem solving. I am not assuming that "problem solving" is simply seen as being about solving problems but it can also be about learning to solve problems.

3.2 Mathematical thinking

There is a lack of distinction in the literature between problem solving and mathematical thinking, with the two terms often used synonymously. An exception to this however is Schoenfeld (1992; 1994), who suggests that mathematical thinking involves:

- developing a mathematical point of view - valuing the process of mathematization (defined in Romberg (1994) from Freudenthal) and abstraction and having the predilection to apply them,
- developing competence with tools of the trade and using those tools in the service of the goal or understanding structure- mathematical sense making.

In Schoenfeld's terms problem solving is part of mathematical thinking. This gives mathematical thinking an overarching role in which problem solving skills are necessary and could be described as "the tools of the trade". One concern I have with this model is that it is not articulating the complexity of what he is describing as problem solving. I will refer to these two points as "mathematical literacy" and reserve "mathematical thinking" to describe the heuristics which underpin mathematical problem solving. That is, to use mathematical thinking as a term which covers the specific mathematical skills we engage with when we problem solve. In this framework mathematical thinking would encompass some of the ideas suggested by Mason and Davis (1991), Ramsey (2004) and Polya (1957) Carpenter, Ansell et al. (1993) including:

- Specialising (specific action that comes out of the problem – doping a particular thing to help to simplify or trying special cases – e.g. paper folding)
- Generalising (as identifying patterns –general or common patterns – formula – looking for an essential shape or form)
- Using analogy (looking at other problems that may have similar structures or develop similar ideas)

- Visualising (using pictures to represent or explain mathematical problem situations or their solutions)
- Identifying the particular
- Modelling
- Decomposing

There is still some work to do in identifying different aspects of mathematical thinking. Not all the strategies we have identified have a similar feel to them. Currently it seems easier to implement a developmental schema for some than for others. We are will underway in writing materials that support the development of skills such as "generalising" and "being systematic" but other skills, such as modelling and decomposing, are proving more difficult. It may therefore be necessary to subdivide mathematical thinking into more specific and more general (less well defined) skills. These and many other elements within our mathematical thinking list require more research.

3.3 Implications for teaching for enrichment

I have discussed above the curriculum content associated with mathematical enrichment in terms of the two aspects of mathematical thinking and problem solving. For this content to have meaning, the learning (and teaching) environment needs to encourage effective use of the resources so that pupils develop the necessary skills, strategies and competence to tackle problems and use underpinning thinking skills effectively. This has implications for the second thread of mathematical enrichment – that of the teaching approach adopted.

This approach reflects constructivist views of learning through social interaction. This construction of knowledge implies that learning builds upon the previous knowledge of the student and their interaction with resources and interaction with members of their community of practice.

The importance of social interaction during successful problem solving is not normally associated with constructivist views of learning, although von Glaserfeld (1995) would deny the correctness of this claim. In itself, the structure of a "good" problem, where students are required to interact and build solution schemas, revisiting and revising ideas, links closely with building on prior knowledge and the constructing of mental patterns associated with a rationalist view of knowledge.

What emerges from the literature, is that students' expectations of mathematics focus on activities that value procedures and accuracy, that are led by the teacher and that do not involve social activity. As they value relatively highly the views of their peers (Adhami, Johnson et al. 1995) social interaction dominated by the teacher is likely to result in limited learning gains compared to carefully constructed learning opportunities where pupils feel autonomous and independent and teachers act as guides. The problem here is how to move away from the norm of teacher-centred classroom practice. To be successful, problem solvers need to be creative, confident and autonomous (Mason and Davis 1991; Pape, Bell et al. 2003).

"In sum, novel work stretches the limits of classroom management and intensifies the complexity of the teacher's task of orchestrating classroom events" Page 174 (Doyle 1988).

What I am suggesting is that the teacher has a key role to play in offering the right material at the right time in the right classroom environment. The classroom environment should facilitate constructivist approaches through social interaction of the pupils and teachers. The teacher's role is to offer suitable tasks, create an atmosphere where students are not passive and use interventions that do not lead but draw mathematics from the students by making mathematical connections and help them to fill knowledge gaps.

This places an emphasis on teaching which stresses:

- non-assertive mediation,
- group work, discussion, communicating …,
- varied solutions and different approaches being valued and utilised,
- exploration, making mathematical connections, extending boundaries, celebrating ideas not simply answers, flexibility… ,

- acknowledgment that maths is hard but success is all the more enjoyable when a hurdle is overcome.

3.4 Enrichment

In terms of content I am therefore proposing a framework for enrichment which comprises two factors, content and teaching. In terms of content problem solving covers the generic range of skills, which have applicability within and beyond the mathematics curriculum and which describe the key elements in the process of problem solving. Mathematical thinking relates to specific mathematical skills students need to draw on in order to problem solve effectively (See Figure 1). Hence, if you were teaching *about* problem solving, it would not be enough to learn the generic problem solving skills but you would also need the mathematical thinking toolbox, without which problem solvers have no skills to apply to the problem solving process.

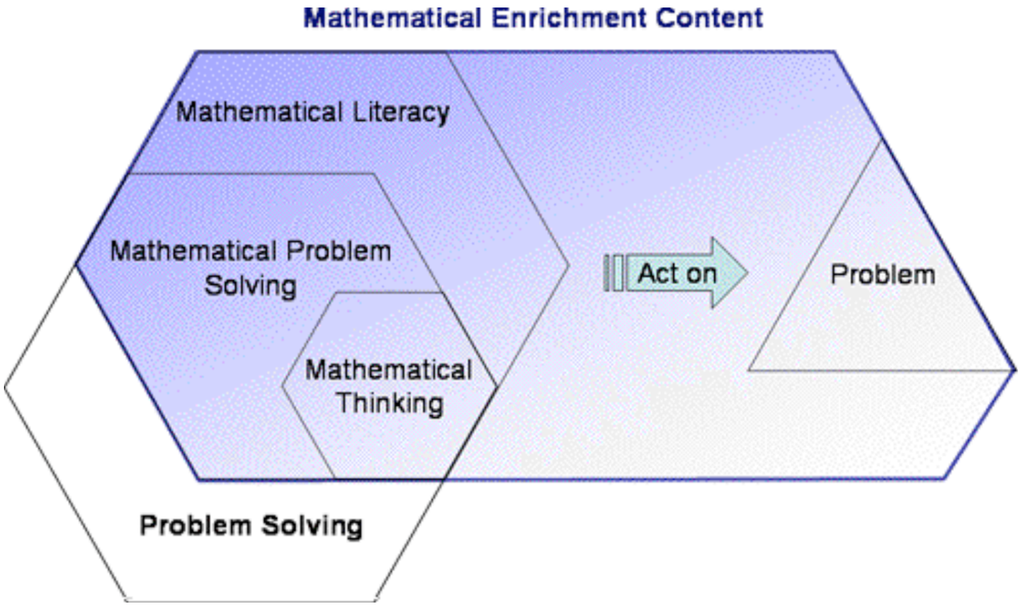

Figure 1

The elements in the figure are then acted on and act on the teacher and the problem solver.

An enrichment curriculum therefore has three content threads and a particular view of teaching and learning that supports problem solving:

Content:

- Engaging problems which:
- develop and use problem solving strategies,
- encourage mathematical thinking.

A teaching approach which encourages:

- an open and flexible environment where we encourage:
- group work,
- exploration,
- mathematical communication,
- the valuing and utilisation of difference as a teaching tool,
- the acknowledgment that mathematics is often hard.

Enrichment is not simply learning facts and demonstrating skills. Mathematical skills and knowledge can be a precursors to, and also outcomes of, an enrichment curriculum (needs driven learning). See Figure 2.

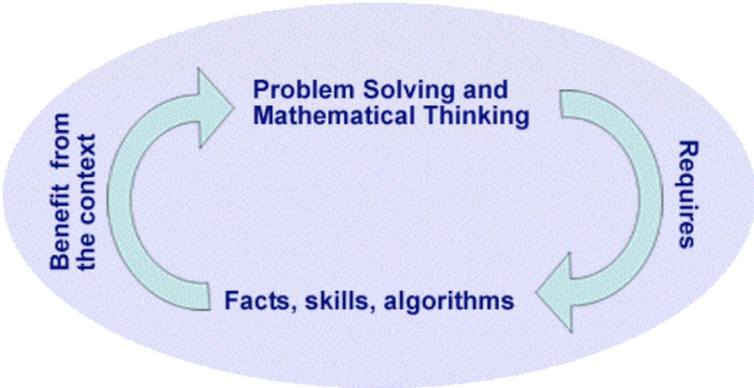

Figure 2

The aim of an enrichment curriculum is to support:

- a problem solving approach (either through, about or for problem solving) that encompasses the four element model,
- improving pupil attitudes
- a growing appreciation of mathematics as a discipline
- the development of conceptual structures that support mathematical understanding and thinking

Enrichment therefore represents an open and flexible approach to teaching mathematics which encourages experimentation and communication

4. Implications for Implementation – is for the most able?

There is nothing above that either implies enrichment should be an add-on or reserved for the special lesson, nor does there appear to be any reason to believe its exclusivity to pupils of higher ability. There is also a growing body of evidence that problem solving approaches and mathematical thinking offered can benefit all pupils (e.g. Schoenfeld 1994; Renzulli and Reis 1999; Landau, Weissler et al. 2001) including low-attaining ones (Watson 2001; Watson 2001). This is also supported by our own work with teachers and pupils when using materials from the NRICH site (www.nrich.maths.org). therefore, if problem solving is seen as a fundamental constituent of enrichment then at the very least this aspect of enrichment can be shown to benefit everyone.

5. Conclusion

In this paper I have therefore attempted to make a case for saying that enrichment should pervade all aspects of our teaching and pupils' learning, whatever the ability of the pupil. It reflects a view of mathematics as a problem solving subject. It can facilitate developing high level mathematical problem solving and thing skills in the most able whilst offering opportunities for everyone to engage at an appropriate level. This does place high demands on teachers but in many ways such an approach, given appropriate resources, may ease the burdens of multiple and different activities going on in any classroom because of the different needs of different learners.

To implement such a view in a practicable way, we need to develop appropriate resources. This is part of the work currently being undertaken by the NRICH project. We have produced enrichment "trails" that aim to support pupils with developing thinking skills such as "being systematic", "generalising", "visualising" and "using analogy" as well as trails based on learning through problem solving, such as "finding areas of triangles".

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