Summing to One

First I shall prove a famous lemma.

Lemma: $\log_a xy = \log_a x + \log_a y$

Proof

Let $a^n=x$ and $a^m=y$

Therefore, $\log_a x = n$ and $\log_a y = m$.

$\Longrightarrow \log_a xy = \log_a (a^n * a^m)$

$\Longrightarrow \log_a xy = \log_a a^{(m+n)}$

$\Longrightarrow \log_a xy = (m+n)\log_a a$

$\Longrightarrow \log_a xy = m+n$

$\Longrightarrow \log_a xy = \log_a x + \log_a y$

This lemma can be generalized to any number of variables as multiplication is associative.

Part 1

$\log_{81} 81 = \log_{81} 3*3*3*3$

$\Longrightarrow 1 = \log_{81} 3 + \log_{81} 3 + \log_{81} 3 + \log_{81} 3$

Hence 4 $\log_{81} 3$ are required to make 1.

Part II

$\log_6 x + \log_6 y = 1$
$\Longrightarrow \log_6 xy = \log_6 6$
$\Longrightarrow$ xy=6 (Eq I)
From (Eq I) all possible ways to do so are pairs of prime factors of 6 such that their product is 6.

So, the ways to do this are (6,1),(3,2),(2,3)&(1,6).

Similarly, it can be generalized that all possible values for the equation $\log_a x + \log_a y = 1$
are pairs of prime factors of a such that their product is a just by replacing 6 with a.

For $\log_{12} x + \log_{12} y = 1$
the values of x&y are
(1,12),(2,6),(3,4),(4,3),(6,2)&(12,1).

For $\log_{24} x + \log_{24} y = 1$
the values of x&y are
(1,24),(2,12),(3,8),(4,6),(6,4),(8,3),(12,2)&(24,1).