# 2010 STEP I Q3

$$\sin(x+y) - \sin(x-y) = \cancel{\sin x \cos y} + \cos x \sin y - (\cancel{\sin x \cos y} - \cos x \sin y)$$ ← using sin addition formula

$$= \underline{2\cos x \sin y}.$$

Let $A = x+y$, $B = x-y$. Then $x = \frac{1}{2}(A+B)$, $y = \frac{1}{2}(A-B)$, and from the above we have

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$ as required.

---

Similarly, we have

$$\cos(x+y) - \cos(x-y) = \cancel{\cos x \cos y} - \sin x \sin y - (\cancel{\cos x \cos y} + \sin x \sin y)$$

$$= -2\sin x \sin y.$$

Then if $A$ and $B$ are as before, we have

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

---

Note that the points $P, Q, R$ and $S$ all lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Knowing this is not necessary to solve the question, but it may help to understand what is going on geometrically.

The lines $PQ$ and $SR$ are parallel if and only if their gradients are equal, ie.

$$\Leftrightarrow \frac{b\sin q - b\sin p}{a\cos q - a\cos p} = \frac{b\sin s - b\sin r}{a\cos s - a\cos r}$$ ← these gradients are non-infinite, since the lines are not vertical.

$$\Leftrightarrow \frac{\sin q - \sin p}{\cos q - \cos p} = \frac{\sin s - \sin r}{\cos s - \cos r}$$

$$\Leftrightarrow \frac{2\cos\frac{1}{2}(q+p)\cancel{\sin\frac{1}{2}(q-p)}}{-2\sin\frac{1}{2}(q+p)\cancel{\sin\frac{1}{2}(q-p)}} = \frac{2\cos\frac{1}{2}(s+r)\cancel{\sin\frac{1}{2}(s-r)}}{-2\sin\frac{1}{2}(s+r)\cancel{\sin\frac{1}{2}(s-r)}}$$

$$\iff \frac{\cos\frac{1}{2}(q+p)}{\sin\frac{1}{2}(q+p)} = \frac{\cos\frac{1}{2}(s+r)}{\sin\frac{1}{2}(s+r)}$$

$$\iff \cot\tfrac{1}{2}(q+p) = \cot\tfrac{1}{2}(s+r)$$

$\iff \frac{1}{2}(q+p) = \frac{1}{2}(s+r) + k\pi$, $k$ some integer $\leftarrow$ since $\cot x = \frac{1}{\tan x}$ has period $\pi$.

$\iff q+p = s+r+2k\pi$, $k$ some integer

$\iff r+s-p-q = 2n\pi$, $n$ some integer.

Now $0 \leq p < q < r < s < 2\pi$, so $r+s-p-q > 0$,

and also $r+s-p-q < 2\pi + 2\pi - 0 - 0 = 4\pi$, so we must have $n=1$, and

$\underline{r+s-p-q = 2\pi}$, as required.