The Mathematics Educator
2003, Vol.7, No.2, 55 - 70

Mathematics as Human Activity: A Different Handshakes Problem

Tim Rowland
University of Cambridge, United Kingdom

Abstract: *The core of this paper consists of a reflective account of the means a solution of a particular problem was obtained. The problem solving literature now includes a number of books that commend such reflective activity, and sest out particular frameworks encompassing heuristics for arriving at solutions. The account in this paper is largely descriptive in the sense that it does not claim to offer a heuristic menu for others to use. It does, however, contrast successful deductive and inductive approaches to solving the same problem, and is intended to encourage students and other problem solvers to maintain reflective awareness of the possibilities available to them as they work on a problem.*

Introduction

The learning of mathematics entails the possibility of the acquisition of several different kinds of knowledge. These include knowledge not only of facts, procedures and concepts, but also of problem solving strategies. Our understanding of the repertoire of such strategies, or heuristics, for problem solving has been enhanced by a number of classic works, such as Polya (1945/1990), Mason, Burton, and Stacey (1982), Burton (1984) and Schoenfeld (1985). It is a moot question as to whether or not much of the literature on mathematical problem solving is *de*scriptive or *pr*escriptive; that is, does it just set out the strategies that successful problem solvers are frequently found to adopt, or is there a suggestion that students can learn (as an outcome of teaching) some of these strategies, *consciously* apply them in problem situations, and become more successful problem solvers as a result? Much of this paper is of the first kind, written in the spirit of description, with a number of possible audiences in mind. It is, in no small measure, a kind of 'wish you were here' postcard, because a good problem solving experience is frequently one that is best shared with someone else.

This paper was also written with a group of my students in mind, as I explain later, to demonstrate how one can be aware of one's own processes of problem solving, and of various 'stages' of solution. It is possible to note the good moments, and what seemed to make them possible, along with the frustrations of being 'stuck', of faulty reasoning and questionable strategy. Moreover, it is possible to do these things in quite a detached way, as if one were observing someone other than oneself. In reflecting on action in this way, one is able to entertain choices about what to do next, and so to exercise control over subsequent action (Mason, 1987).

In a concluding section, I shall reconsider prescriptive and descriptive aspects of problem solving heuristics, and attempt to distinguish and legitimate a certain *genre* of mathematical writing to which much of this paper belongs. The account which follows is offered, here, to mark the 20th anniversary of the publication of Mason, Burton and Stacey (1982), an event that was celebrated in May 2002 in the presence of the three authors at the conference *Mathematics Education in a Knowledge-Based Era* at Nanyang Technological University, Singapore.

Methodology

It is no mere coincidence that the methodological paradigm in which this paper is located – 'researching from the inside' - also owes much to John Mason, and to Joy Davis (e.g., Mason, 1994). The approach to research and to personal/professional development (as a researcher, a teacher, a mathematician, a problem solver …) centres on the extending and sharing of awareness by telling 'stories', and by encouraging others to tell theirs. As Feyerebend noted, "All you can do, if you really want to be truthful, is to tell a story" (quoted by Mason, 1994, p. 177). The two stories in this paper are somewhat different in kind. The first is an account of a personal intellectual journey over a period of about 24 hours. This began with receiving a motivating (for me) problem, and concluded with a solution. This inner 'journey', from problem to solution, is set against the backdrop of an actual journey – a 20-mile walk following the course of a river. I attempt to justify the combination of both journeys in the same account towards the end of this paper. The second story is the description of a 'lesson' with a group of undergraduate students, working on the same problem. The lesson lasted just over an hour, and while the second story is therefore somewhat shorter than the first, I regard it as of equal importance and with equal potential for learning, for myself and (perhaps) for others. Wheeler (cited in Mason, 1994) advocated the professional exchange of such incidents, episodes or lessons. Earlier, in the United Kingdom, a Mathematical Association working group abstracted a model for developing teaching, known as *the anecdoting process* (Jaworski, 1991). This was based on teachers' stories or anecdotes used to promote the raising of issues and addressing of critical questions relating to teaching. In this way, telling and reflecting on stories such as those in this paper has the potential to "inform perceptions, observations, analysis and theorising". (Mason, 1994, p. 179)

The problem

It began with an email on a Thursday in October, from a friend, Tony, a mathematician turned musician.

> A man and his partner invite five other couples to dinner. As they enter the house, and accept pre-dinner drinks, the hosts and their guests shake hands

> with each other. Not all of them do, for various irrelevant reasons. However, naturally, no guest shakes hands with his or her partner. At dinner, the host brings up the subject of the handshakes. Going round the table and asking everybody else in turn how many hands he or she shook, he finds that he obtains 11 different answers. How many hands did his partner shake?

"A man and his partner invite 5 other couples to dinner". It just so happened that this described a situation (admittedly idealised for the sake of a neat statement of the problem) in which I would participate in Oxford in two weeks time. This was clever on Tony's part, yet it would be wrong to say that my motivation was enhanced by the belief that this was, in some sense, 'real' problem solving. For me, the problem is a 'nice' one irrespective of its real-life relevance. Parenthetically, I think we have a problem with this issue in UK elementary schools at present: the tendency to appeal to real-life contexts for the purpose of motivating students seems to neglect the motivational power of mathematics itself.

Tony's email mentioned that he'd be away for the next week - ruling out any further clarification of the problem. My first reaction was that the problem could not be solved. Twelve people had shaken hands a number of times, but how could one possibly distinguish among them? If they were permuted, the story would be much the same. True, there was no information about the host's handshakes, but nothing to isolate any of the remaining 11. Only two thoughts prevented me from rejecting the problem out of hand at this point. First, I knew that my friend would be unlikely to pose pointless problems. Secondly, if there was a solution, I wanted to be able to show him that I had made some attempt at solving it!

Understanding the problem

Friday was a busy day, and I was unable to give any further sustained thought to the problem, although the conviction that it had a solution, and a unique one at that, strengthened. It just occurred to me that the size of the problem could be reduced by trying the same problem with, say, not five pairs of guests, but two. As Polya advises, "If you cannot solve the proposed problem … can you imagine a more accessible related problem?" (Polya, 1990, p. 114). I regularly recommend such a strategy to my students, but (for some forgotten reason) decided that there was something special about the "5" in this problem to make possible a solution, and a unique one at that. This decision turned out to be a *big* mistake!

During Friday evening two lines of attack occurred to me. Both were ultimately fruitless. First, I saw the solution as a 'graph': a set of nodes (points) - the guests - and edges (lines) joining those who had shaken hands. I called to mind the classic 'handshakes problem', where n people all shake hands with each other. The total number of handshakes is $\frac{1}{2}n(n-1)$, which is 66 when $n = 12$. Now[1]

> since nobody shakes hands with themselves or their partner, the maximum number for any one person is 10. Thus the host's guests and his partner cover the range 0 to 10. This amounts to 55 handshakes.

But this means that the host must have shaken 11 hands. Impossible! Then I realised that the sum of the number of hands shaken by each person is *twice* the number of handshakes. This is at most 110, and will be even in any case, whether or not all hands have been shaken by everyone. Fifty-five have been accounted for, excluding the host, therefore:

> The number for the host is unknown, but must be odd, since handshakes are symmetrical (if A shakes hands with B then B shakes hands with A). Thus the host shakes 1, 3, 5, 7 or 9 hands. In particular, he does not shake all possible hands.

I actually marked 12 points in circular formation, like the numerals on a clock face. Pairs of adjacent numerals 1-2, 3-4, and so on, would be partners. I marked a few of them M1, W1, and so on. I had to be careful not to join any M*k* to W*k*, since people don't shake their partners' hand. I started to join some of the points in keeping with the information. I would make M1 the host. Suppose M3 is the guest who shakes just one hand. But *which* hand? Did it matter? Better not join him to W1? But what if it is actually necessary to do so to construct the solution? I drew in a few lines speculatively, and then abandoned the diagram. How could I possibly know which points to join?

Secondly, I envisaged a 12x12 grid, with the 12 people along each row and column heading. I would enter a '1' if they had shaken hands. It would have to be symmetrical. The sums of the rows and columns (apart from the host) would give 0 to 11. I might do this on an Excel spreadsheet for two reasons. It would facilitate adding the rows/columns. It would also be amenable to revision, to adjustment until I got it right. I could also do it with pegs on pegboard (though they were not to hand) or counters on squared paper. By now, it was too late into the evening to contemplate doing either.

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[1] The indented sections here and later are quotations from a succinct statement of my 'solution' that I wrote first.

I woke early - about 6am - on Saturday, and started thinking about the grid formation and how one might set about assigning the 1s to the cells. I envisaged the row of the person who shakes 10 hands (I pictured him as a rather hearty male!) It would have ten 1s and two blanks. Which cells would be blank? Those corresponding to that person and their partner! This led to the following realisation:

> One guest (call them A) shakes all 10 possible hands. Now there is also just one person who shakes no hands. But everyone except A and his partner shakes hands with A, so it is A's partner who shakes no hands.

This was a breakthrough of a kind, yet it seemed to be only the tip of the iceberg. There remained five more couples, about which I seemed unable to say anything as yet.

A walk by the river

Most unusually, I had planned to spend Saturday walking, following the river from Ely to Cambridge. I packed a flask of coffee and the usual bad-weather contingencies, together with a few sheets of writing paper. I could see no way forward with the problem other than with a grid of some kind, making the data 'work', showing which pairs had actually shaken hands. That way a solution might emerge, although I might not be confident that it was the *only* solution.

I was standing on the Ely riverside before 09:30. It was the most perfect late autumn day. As I walked along the flood bank to the west of the river, my mind ranged over all kinds of topics, most of them nothing to do with mathematics, but my thoughts returned seemingly involuntarily to the handshakes problem from time to time. I did want to make some progress with it, yet I was also aware of the whole day stretching ahead of me. Crucially, I believe, I felt under no pressure of time. My mind was freed up to think about it as and when I chose to.

I recapped the argument about the person who shakes no hands being the partner of the one who shook 10. I said it out loud to myself and a few swans at the river's edge.

> OK then, what about the person who shook 9 hands?

As I walked, I sensed one difficulty that seemed to be insurmountable, that of retaining everything that I might be able to deduce in my memory. In effect, I was trying to place the 1s in a 12x12 grid that I was carrying in my head. It was all too much. It was precisely this difficulty that led to my making an important shift in my strategy. In the grid, I knew all there was to know about the row and column of the

10-handshake person and that of his or her partner. I gave them the first two rows and columns. What would it be like if I simply *removed* these two rows and columns, as if I'd cut them off with a pair of scissors, as if they'd never come to the party, in fact? That would leave only 5 pairs of couples and their handshakes, and I would have less to retain in my mind, with 100 cells instead of 144. But the following adjustment would have to be made. In the original problem there was a person who had shaken just one hand, and this hand must have been A's. With A taken out of the picture, this person will have shaken no hands. Likewise, the number of handshakes of the 10 people remaining must *all* be reduced by 1. The range was from 1 to 9 (with the host unknown), but now it becomes 0 to 8.

> If we now remove A and A's partner from the scenario, as if they'd never come, we are then left with the host, his partner and four pairs of guests. Now the partner and the guests had originally shaken between 1 and 9 hands. Taking A (who shook all hands) out of the frame and A's partner (who shook none), the remaining 9 (not the host) have shaken between 0 and 8 hands. The host, also reduced by 1, has shaken an even number.

Reasoning as I had before, what can I say about B, the person who shakes 8 hands? Incidentally, I had left the host and his partner still in my reduced scenario, although I had not eliminated the possibility that A is the host's partner! In fact, I didn't think to do so until later that evening, when I wrote a succinct account of my solution, but I'll insert it at this point:

> Note that the host's partner cannot be A (10 shakes) since, as we have seen, A's partner shakes no hands, whereas the host must have shaken an odd number. Neither can the host's partner be A's partner (0 shakes) since A would then be the host, and the host has not declared how many hands he shook.

I had reached the crossing of a tributary of the river at that point. I told myself that B shook everyone's hand except his or her partner's, giving 9 possibilities for the 8 handshakes, so the allocation of B's handshakes could not be resolved by such reasoning. I decided that the situation for B was logically different from that for A. I was disappointed, and my thoughts wandered off in some other direction. I briefly returned to and recapped the same argument a couple of times in the next hour or so, with renewed disappointment. I had the writing paper in my bag: perhaps it would help if I drew the grid when I stopped for lunch.

Of course, my reasoning about B was faulty. It was about 12:30 when I realised that B could not shake his partner's hand or his own, so he must shake all the remaining 8. Once again, it followed that B's partner could not shake any.

> The same argument then establishes that the person, B say, with 8 shakes (9 in the original scenario) is the partner of the one with 0 shakes (originally 1), and neither is the partner of the host.

At that moment I knew that I had solved Tony's problem! That is not to say that I knew the answer, but I *did* know that I could arrive at it by careful iteration of the same argument, removing one couple at a time until I was left with a manageable, even trivial, problem.

> We then take B and his/her partner out, reduce all handshakes by 1 again, being left with the host, together with his partner and three pairs of guests who have shaken between 0 and 6 hands. The host, also reduced by 1, has shaken an odd number.

I kept tally on my fingers as I walked, to make sure I would make no careless errors, as I spoke aloud. "Two pairs of guests, 0 to 4, host even. One pair of guests, 0 to 2, host odd." I couldn't trust myself to remove all the guests in one final iteration! Instead, I argued that the visitors accounted for 2 and 0, as before, and if the host was odd, then he (like his partner) must have shaken 1 hand. I sketched a mental diagram of how this could happen; one guest shakes hands with the host and his partner. After four reduction stages, I had removed 4 handshakes for each of the remaining participants, so in the original scenario, the host and his partner had shaken 5 hands.

Iterating twice more (and removing C, D and their partners), we are left with the host, together with his partner and one pair of guests who have shaken 0, 1 and 2 hands, the host having shaken an odd number, which must in this case be 1 (because he does not shake his own hand or his partner's). The usual argument establishes that the guests account for 0 and 2, so the host and his partner have each shaken one hand - E's, in fact. In the four stages of reduction, there is a reduction of one handshake for each of the remaining participants, so both the host and his partner in fact shook 5 hands in the original scenario.

I then wondered why Tony's problem had specified only the host's partner, and not also the host himself, since the number of his handshakes could also be determined. Perhaps it was to heighten my original sense that the problem could not be uniquely solved, because (as I mistakenly supposed) the host's partner could not be distinguished from the guests.

Over the course of the rest of the walk, I added various additional observations, such as:

> the number of handshakes of each couple sum to 10 i.e. 0-10. 1-9 etc (and 5-5 in particular)

I also realised that it may indeed have been beneficial to follow up my early instinct to solve first a similar problem with a smaller number of pairs of guests. There was nothing special about specifying 5 pairs of guests.

> the solution generalises to a similar problem with any number of couples e.g. with 10 couples, the pairings are 0-18, 1-17 etc with 9-9 for the host and his partner.

I told myself that considering fewer guests might not have been productive, however, since I would have solved the simpler problem with a nodes-and-edges graph, and might not have been able to generalise it. (Once again I was wrong, as the next section will show.)

I gave some thought as to how the handshakes could actually be arranged to fit the data - I had in mind an algorithm, a procedure to specify how each guest would know which hands to shake. I had not resolved it by the end of the walk, and I was content to leave it that way - for the time being, at least.

The same evening I wrote a succinct account of my solution, in case I forgot it (although I now think it unlikely that I would have done so, not for some weeks at least). At the end of my account I added the retrospective observations:

> the solution is not unique - there is symmetry within the set of pairs of guests, nor can one distinguish between partners. The solution for the *hosts* is unique, however, because (a) their numbers of handshakes are the same (b) we happen to be told that the man makes the enquiry about handshakes.
>
> It would have been possible, if a little obscure, to perform one more iteration in the argument, leaving just the host (handshakes unknown) and one person - his partner - with 0 shakes. In this ultimately reduced scenario, the number of shakes for the host is clearly also 0. In the five stages of reduction, there has been a reduction of one handshake for each of them, so both the host and his partner in fact shook 5 hands in the original scenario.

The following day, Sunday, I began to write this extended account of the process of my solution for the benefit, in the first instance, of some new undergraduate students with whom I'd just begun a problem solving course. The course is placed in the very first term of a joint honours degree in mathematics and education, and aims to direct their attention towards processes as opposed to the 'products' (theorems and the like) that their mathematical experience to date has tended to prioritise. These students were expected, as course assignments, to write accounts of the processes whereby they arrive at their own solutions to various problems. I could offer them my writing on the 'handshakes' problem as one example of such an account.

Working with the class

The next day, Monday, I had my second session with this same undergraduate class. A number of them were missing, with various ailments. I took this as my excuse to introduce Tony's problem, instead of reviewing the previous week's 'homework' problem as planned. I did this with some trepidation. Since it had taken me some time to solve the problem, I suspected that it would be unresolved by the end of the 90-minute session, and they would have to continue working on it afterwards. If very little progress had been made in the 90 minutes, I considered offering them, as a 'hint' the idea of removing the 10-handshake person and his/her partner, as I had done. In fact, that proved to be entirely unnecessary!

More often than not, having introduced a new problem to the class, I ask them to work on it for half an hour or so in twos or threes to begin with, before reviewing progress in 'plenary', with the whole class. On this occasion I suggested that we discuss and work on it together from the outset, and they seemed happy enough with this. It can be a very stimulating way of working (especially for their tutor!) but one has to be very careful to orchestrate the discussion so that as many as possible contribute. In fact, with all such classes, I introduce the following advice for them to consider:

> let it be the group task to encourage those who are *unsure* to be the ones to speak first [...] every utterance is treated as a *modifiable conjecture*. (Mason, 1988, p. 9)

Their first reactions to the problem echoed my own: Jenny said that it sounded impossible, while Katrina recalled the 'handshakes theorem' from school: $\frac{1}{2}n(n-1)$ handshakes when n people all shake hands with each other. She had asked herself Polya's 'devising a plan' question: "Do you know a related problem?" (Polya, 1990, p. 9). Polya himself, however, pointed out that this line of approach does not always "work", and it was apparent to Katrina herself that the conditions of the two problems were very different. I wanted to see what would happen if I reminded them of Polya's 'planning' maxim from the previous week's session: "If you cannot solve the proposed problem, try to solve first some related problem" (*ibid*., p. 10). What would such a problem be like in this case? Jenny suggested that we think about, not five pairs of guests, but one. This struck me as quite a bold simplification, and I wondered how much insight it would give into the much more complex, original problem. Well, there would now be 4 people present at the party, and since people cannot shake their own hands or their partner's, the maximum number of shakes for each person is 2. When the host asked the three others about their number of handshakes, these would have to be 0, 1 and 2. I asked whether we could solve that problem. Once the symmetry of a handshake was made clear, they had no difficulty in finding a solution, as I had done walking along the river bank two days earlier. And what if there were two pairs of guests? What would be the different numbers of handshakes? Angela explained why it would have to be 0 to 4, since 5 was impossible. I sketched three pairs of dots on the whiteboard, put brackets around one of them, which was understood to be the host (a notation introduced by Jenny). Could we draw lines between 5 of the dots to show 1, 2, 3, 4 handshakes, with a sixth dot left isolated? It was Samantha who first found a solution, and sketched it on the board. The host and his partner shook 2 hands each, one pair of guests shook 3 and 1, the other pair 4 and 0.

It seemed to me that there were enough 'data' here to offer guidance in the solution of the more complex cases in general and, the case with six couples, in particular. What is interesting is that starting with small numbers of guests, as Jenny had suggested, opened up the possibility of a very different solution heuristic from that which I had adopted earlier, working on my own. I wanted to invoke, or stimulate, their powers of *inductive* reasoning (Rowland, 1999), with predictions about other particular cases and conjectures about the general case. With hindsight, I wish that I had asked the class what *they* considered to be the best way forward at this stage. If I had, it is possible that someone would have suggested solving the 'next' case, with three pairs of guests. Of course, I don't know, but I did want to avoid getting bogged down in the details of these progressively more complex cases. Instead, I wanted them to draw on what they had found already.

It's interesting to reflect on the seeming possibility that I exerted more control over their solution strategies when we worked as a whole class. Normally, when I leave the students to begin work on a problem is twos or threes, I tend merely to observe what each group is doing so that I can call on a handful of students to report on different approaches in the class discussion later. Working as a whole class from the outset, I'm much more reluctant to let them go down (what I perceive to be) blind alleys. When I foresee a 'nice' way forward from some point in the solution process – use of inductive reasoning in this case – I wonder if I tend to 'steer' them in that direction? When I do, they tend not to complain, perhaps because their experience of learning mathematics hitherto leads them to expect teachers to behave that way.

Perhaps they feel more secure, more confident of finding a solution if I direct them to some extent. And yet … for me there is a real tension (when working with the whole class) between trying to ensure that they experience a range of successful and elegant procedures on the one hand and giving them complete freedom to determine (or not) a solution on the other. This tension is related to Polya's (1990) remark that the teacher of problem solving has two aims, namely for his (*sic*) students to solve the problem in hand *and* to equip them to solve future problems by themselves. I must remember to talk to them about that.

And so it was that I asked what they could 'see' in the solutions that they had found for 4 people, and for 6, that might guide us in finding a solution for 8. Jenny tentatively observed that the host and his partner each shook 1 hand in the first case, and 2 in the second. This might just be a "coincidence", she said, but maybe they would shake 3 each if there were 3 pairs of guests. Angela's observation was less tentative, and took Jenny's one step further forwards: the *sums* of handshakes for each couple was the same - 2 in the first case, 4 in the second. Perhaps it would be 6 if there were 3 pairs of guests, she predicted.

So we were drawn by Angela's prediction to consider the 'next' case, with 4 couples. This time I drew 4 pairs of dots on the board, in 4 rows, writing 3-3, 4-2, 5-1, 6-0 against the dots in each pair, in keeping with Angela's proposal. Could they now draw in the handshakes, as the edges on a graph with 8 vertices, to fit the statement of the problem? I was still cautiously drawing in the first few lines on my own notepaper when Lynsey said it was "logical". She came to the board to explain. I took notes! [The notation which follows is mine, to assist this exposition].

> Call the couples M1-W1 (the hosts), M2-W2, M3-W3 and M4-W4. Now suppose W4 shakes no hands and M4 six. Connect M4 to all the 6 hands 'above' him in the diagram, i.e., M3, W3, M2, W2, M1 and W1. Now W3 has shaken 1 hand, and must shake no others. M3 has also shaken 1 hand, and must shake 4 more - the two pairs above him. Now M2 has shaken 2 hands and must shake no more … and so on. The solution was beautiful both in its certainty and its simplicity. As she returned to her place, Lynsey remarked that you could do that for any number of pairs of guests.

Indeed, you could. For Lynsey, as for me (and others in the class), the solution procedure she had described for four couples was a generic example (e.g., Rowland, 2001) for the other cases. Later, I gave my own version of Lynsey's algorithm for the general case, to myself, as I drove home. The guests arrive in pairs: on arrival, each man shakes hands with his two hosts and those of all *subsequent* arrivals. Whilst the women accept these handshakes (all with men), they initiate none themselves. A more consistent, if less plausible, story, would be that the hosts were absent as the pairs of guests arrived. On arrival, each man shakes hands with all *subsequent* arrivals, including finally those of their hosts, who enter the room when all the guests have arrived. As before, the women accept these handshakes, but initiate none themselves.

It was satisfying to compare the solution we had generated together with the one I had produced on my own. Mine had been purely deductive, an exercise in logic. This one had been inductive, an exercise in conjecture and verification, and finally proof in the form of an algorithm. My own solution shows that the solution later generated by the class is unique (in the sense I have described earlier). Conversely, the class solution specifies the actual pattern of handshakes, which I had not succeeded in finding on my own.

A final thought about my own role in this session is the recognition that, at some point, I became (in my own perception, if not in theirs) just one of the group working on the solution. I think that this happened not long after I had asked them to reflect on the 'data' in the solutions for two and three couples. Lynsey had found a solution 'graph' conforming to Angela's prediction long before I had and her algorithm for the general case marked very significant progress on my own, unaided achievement.

Final remarks

In many respects, there is little that I would wish to add to what I have written above, in the preamble and in the account itself. Being able to work on the problem with others so soon after grappling with it myself was a real bonus. The fact that they were 'my' students meant, perhaps, that they could not easily refuse my 'invitation' to work on it. In the event, their engagement with it far exceeded my expectation, and I was invigorated by the fact that their approach was so different from the one I had taken.

Problem-Solving Heuristics

Both of the approaches to the solution of the problem described in this paper can certainly be analysed from the perspective of Polya's four phases of problem solving (Polya, 1990). There are significant similarities and also some differences between my approach and that taken in the class session. Regarding *Understanding the Problem*, there was some uncertainty in both situations as to whether the data were sufficient to enable a solution of the problem. I drew diagrams (in my head, admittedly), the class introduced notation, though not at the outset. In *Devising a Plan*, we both recognised a related problem, i.e., when each person shakes one hand of every other person. This was useful, though only moderately so. My key insight was the possibility of reducing the number of guests by two (as described in the narrative). For the class, the crucial strategy was to solve a more accessible, related problem, and then to reason inductively. Once these strategies had been decided upon, *Carrying Out the Plan* was relatively straightforward in each situation, although two ‘aha’ moments stand out in both cases. For myself, it was the realisation that the partner of ‘B’ shook no hands in the ‘reduced’ situation; for the class, it Lynsey’s statement of her solution algorithm that was a key moment - certainly for me, her tutor! Finally, *Looking Back* entailed just a few finishing touches in both cases. I embellished both my own solution on arriving home after the walk, and the students’ whilst driving home that day. I hope that at least some of the students also gave it further thought, but I don’t know. Perhaps, for some of them, such reflections may have been prompted when I asked them to read an early version of this paper.

Having said all that, I remain convinced that there was little or no conscious appeal to Polya’s phases (or anyone else’s) *during* the actual process of solution. This is at least consistent with the position taken by Ian Stewart in his Foreword to the Second Edition of *How to Solve It*. Stewart argued, in effect, that these and other, similar heuristics for problem solving are only useful when, in some unconscious, tacit way, one already knows them. In effect, one gets better at problem-solving by solving problems. This is rather like saying that one gets better at running by running, and not by contemplating the component processes that together constitute running. Stewart wrote in terms of the “raw talent” of Mathematical Olympiad competitors, but the conclusion about tacit reference to heuristics would seem to be true more generally. What does seem to be very relevant - and is indeed a major motivation for my writing this paper - is Alan Schoenfeld’s conclusion (quoted by Stewart) that “a large part of what comprises competent problem-solving behaviour consists of the ability to monitor and assess what one does while working problems” (quoted in Polya, 1990, p. xvii).

Context and Genre

I showed an early draft of this paper to a colleague with whom I share the teaching of the problem solving course, and asked her to comment on its suitability for our students and, perhaps, for a wider audience. She is a true ‘critical friend’, and her comments were predictably helpful. Nevertheless, one of her remarks prompted a certain tension in me. This concerned my inclusion (in that version) of certain temporal and spatial details which were very personal to my circumstances as I worked on the problem, such as travelling by train to Ely, morning frost in the grass beside the river, a rest at a pub, and so on. My colleague pointed out that we wouldn’t want our students to include such minutiae in the accounts that they wrote for us to read - we don’t want to know, for example, whether they had tea or coffee at breakfast! I readily agreed, and laughed at my own expense, knowing before I asked her that she might say exactly that. Some of these details have been duly cut in this version of events. And yet … in the text of a lecture given in 1908, reflecting on his insight into a connection between Fuchsian functions and non-Euclidean geometry some 27 years earlier, Henri Poincaré wrote:

> Just at this time, I left Caen where I was living, to go on a geological excursion under the auspices of the School of Mines. […] Having reached Coutances, we entered an omnibus to go some place or other. At the moment when I put my foot on the step, the idea came to me. […] On my return to Caen, for conscience sake, I verified the result at my leisure. (Quoted by Hadamard, 1945, p. 13).

The significance of the omnibus ride had evidently stayed with Poincaré for all those years. But why? It is not unreasonable to suppose that those contextual factors were, for him, the warp into which the weft - the mathematics - was woven. Likewise, I wrote about such *minutiae* in my original account - about early morning frost and 'talking' to ducks on the river - because they actually *matter* to me, because they are the substrate of my recollection of my problem solving experience. If *only* the solution matters, and the means by which I arrived at it, then why mention at all the fact that I solved it whilst walking by a river? I would argue that this aspect, at the very least, of the context matters in this case for two reasons: First, because of the constraints that it imposed, notably lack of access to technology and the necessity to work mentally, without recourse to pen and paper. Secondly, because of the mental freedom that it made possible, by which I mean the prospect of six hours or more in which to think about the problem, free of all other demands. The means by which we solve problems, and whether we succeed in solving them at all, is not independent of such considerations. Moreover, the network of ideas and memories which we assemble in the course of solving a particular problem is more than a set of abstract mathematical units; it has temporal, spatial and emotional components, some of which might stay with us long after we forget our solution of the problem *per se*. In the end, what we present for others to read will depend on the context and the audience. The statements of theorems and presentations of proofs that appear in textbooks and lecture notes are typically stripped of any hint of their genesis. In terms of reporting a knowledge-product, this may not be a problem, but from the perspective of process it is; for, as Brousseau wrote:

> … such a presentation removes all trace of the history of this knowledge, that is, of the succession of difficulties and questions which provoked the

appearance of the fundamental concepts […] It hides the "true" functioning of science … (Brousseau, 1997, p. 21).

There are reasons for this practice, mostly to do with the traditional *genre* of mathematical writing, which presents mathematics as an autonomous system, a passive voice or third person entity as opposed to the product of human activity (Morgan, 2001). Indeed, the research paper *Sur les Fonctions Fuchsiennes* that Poincaré himself published in 1881 makes only passing reference to the connection with non-Euclidean geometry, and, of course, none whatsoever to omnibus rides. I would argue that there must be a place, within mathematics education certainly, for an alternative *genre*, one that liberates the first person voice of those who 'do' the mathematics. Such mathematical texts are likely to be 'stories', narratives which emphasise, rather than suppress, the human dimension, and make available to others not only the ends of mathematical activity, but the means as well.

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Author:

Tim Rowland, University of Cambridge, Cambridge, United Kingdom
tr202@cus.cam.ac.uk